Section 1.1 Q3: For each of the following state the order, whether it is linear/nonlinear, homogeneous ...

a) $u_t - u_{xx} + 1 = 0$

Linear, inhomogeneous, order 2.

b) $u_t - u_{xx} + xu = 0$

Linear, homogeneous, order 2.

c) $u_t - u_{xxt} + uu_x = 0$

Nonlinear (because of third term). Order 3.

d) $u_{tt} - u_{xx} + x^2 = 0$

Linear, inhomogeneous, order 2.

e) $iu_t - u_{xx} + \frac{u}{x} = 0$

Linear, homogeneous, order 2.

f) $u_x (1+u_x^2)^{-1/2} + u_y (1+u_y^2)^{-1/2} = 0$

Nonlinear, order 1.

g) $u_x + e^y u_y = 0$

Linear, homogeneous, order 1.

h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$

Nonlinear (because of third term). Order 4.

Section 1.1 Q4: Show that the difference of two sol's of an inhomogeneous linear eq. $\mathcal{L}u = g$ with the same g is a sol' of $\mathcal{L}u = 0$.

Let v, w be solutions of $\mathcal{L}u = g$. Then

$$\mathcal{L}(v-w) = \mathcal{L}v - \mathcal{L}w = g - g = 0.$$

Section 1.1 Q7: Are the functions $1+x$, $1-x$, $1+x+x^2$ linearly dependent or independent?

We check if we can find a linear combination giving 0:

$$a(1+x) + b(1-x) + c(1+x+x^2) = 0$$

$$\Longrightarrow \quad cx^2 + (a-b+c)x + a+b+c = 0$$

This can hold if and only if the coefficients of $x^2, x, 1$ are all 0:

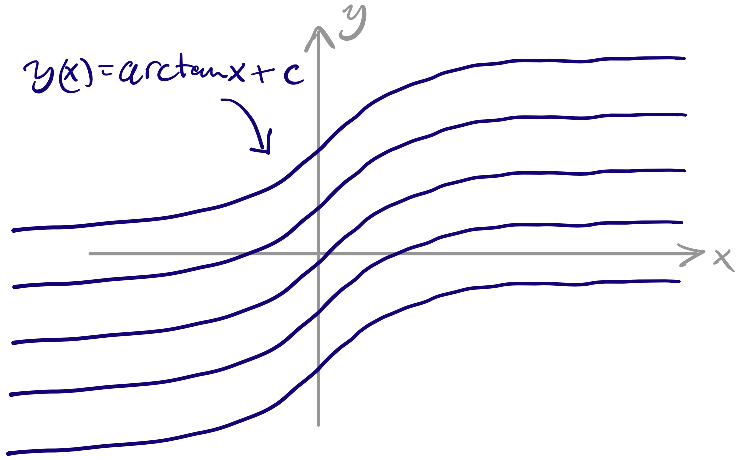
$$\Rightarrow \begin{cases} c = 0 \\ a-b+c = 0 \\ a+b+c = 0 \end{cases} \Longrightarrow \begin{cases} c = 0 \\ a-b = 0 \\ a+b = 0 \end{cases} \Longrightarrow a = b = c = 0$$

This means that the linear combination is trivial, hence these functions are linearly independent.

Section 1.2 Q3: Solve $(1+x^2)u_x + u_y = 0$.

The characteristic curves are given by $\frac{dy}{dx} = \frac{1}{1+x^2}$.
Integrating this, gives $y(x) = \arctan x + c$.
Therefore $u(x,y) = f(y - \arctan x)$ where f is any differentiable function.

The characteristic curves have the form:
(along each of these curves the solution $u(x,y)$ is constant)



Section 1.2 Q4: This is just the chain rule.

Section 1.4 Q4: A rod is heated with $H>0$ along $x \in (\ell/2, \ell)$ and 0 along $x \in (0, \ell/2)$ and temperature fixed at 0 at $x=0$, $x=\ell$. What is the asymptotic temperature profile?

We need to solve $-\partial_{xx} u = f(x) = \begin{cases} 0 & x \in (0, \ell/2) \\ H & x \in (\ell/2, \ell) \end{cases}$

with the boundary conditions: $u(0) = u(\ell) = 0$.

Integrating the eq. we find $-\partial_x u = \begin{cases} a & x \in (0, \ell/2) \\ Hx + c & x \in (\ell/2, \ell) \end{cases}$

Integrating again:

$$-u = \begin{cases} ax + b & x \in (0, \ell/2) \\ \frac{1}{2}Hx^2 + cx + d & x \in (\ell/2, \ell) \end{cases}$$

Now: require $u(0)=0 \Longrightarrow a\cdot 0+b=0 \Longrightarrow b=0$

$u(\ell)=0 \Longrightarrow \frac{1}{2}H\ell^2+c\ell+d=0$

u continuous at $\frac{\ell}{2} \Longrightarrow a\frac{\ell}{2}+b=\frac{1}{2}H\left(\frac{\ell}{2}\right)^2+c\frac{\ell}{2}+d$

$\partial_x u$ continuous at $\frac{\ell}{2} \Longrightarrow a=H\frac{\ell}{2}+c$

We have 4 eq. for the 4 unknowns a, b, c, d.
Solving, leads us to:

$b=0 \qquad d=\frac{1}{8}H\ell^2 \qquad a=-\frac{1}{8}H\ell \qquad c=-\frac{5}{8}H\ell$

So we get

$$u(x)=\begin{cases}\frac{1}{8}H\ell x & x\in(0,\ell/2)\\ -\frac{1}{2}Hx^2+\frac{5}{8}H\ell x-\frac{1}{8}H\ell^2 & x\in(\ell/2,\ell)\end{cases}$$

and

$$u'(x)=\begin{cases}\frac{1}{8}H\ell\\ -Hx+\frac{5}{8}H\ell\end{cases}$$

$u'(x)=0 \Longleftrightarrow Hx=\frac{5}{8}H\ell \Longleftrightarrow x=\frac{5}{8}\ell$

$$u\left(\tfrac{5}{8}\ell\right)=-\frac{1}{2}H\left(\frac{5}{8}\ell\right)^2+\left(\frac{5}{8}H\ell\right)\left(\frac{5}{8}\ell\right)-\frac{1}{8}H\ell^2$$

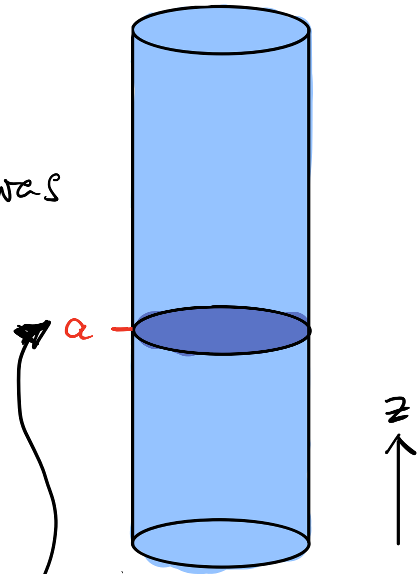
$$=-\frac{25}{128}H\ell^2+\frac{25}{64}H\ell^2-\frac{1}{8}H\ell^2$$

$$=\left(-\frac{25}{128}+\frac{50}{128}-\frac{16}{128}\right)H\ell^2=\frac{9}{128}H\ell^2$$

Section 1.4 Q5: In Exercise 1.3.4, find the boundary condition if the particles lie above an impermeable horizontal plane $z=a$.

Recall that we have diffusion + constant velocity V downward, so that the eq. was

$$u_t = k u_{zz} + V u_z .$$



Now, we're told that the particles cannot pass through a membrane located at $z=a$

Recall that in 1.3.4 we looked at the "flow in" at $z=z_1$ and the "flow out" at $z=z_0$ and found that:

$$\text{flow out} = k u_z(z_0,t) + V u(z_0,t)$$

Think of $z=a$ as taking the role of $z=z_0$.
So, at $z=a$ the particles are stopped $\Rightarrow$ there's no flow out. Hence $k u_z(a,t) + V u(a,t) = 0$. So the full equation with the boundary condition is:

$$\boxed{\begin{cases} u_t = k u_{zz} + V u_z & -\infty < z < \infty \quad t > 0 \\ k u_z(a,t) + V u(a,t) = 0 & t > 0 \end{cases}}$$