MA 18 MIDTERM SOLUTIONS

(1) For each of the following questions indicate whether it is *true* or *false*. If it is true, *justify*. If it is false – *give a counter example*.

(a) Let $f(x, y)$ be a differentiable function. Suppose the restriction of the function to the curve $x^2 + y^2 = 1$ has a relative maximum at the point $(1, 0)$. Then $\nabla f(1, 0) = (0, 0)$.

(b) Consider the path $\mathbf{c}(t) = (\cos t, \sin t)$. Then $\mathbf{c}(t)$ and $\mathbf{c}'(t)$ are perpendicular.

Solution.

(a) False. Counter example: consider the function $f(x, y) = x$.

(b) True. Can show this either by a direct calculation, or since $\|\mathbf{c}(t)\| = 1$ and therefore $\mathbf{c}(t)$ and $\mathbf{c}'(t)$ are perpendicular.

(2) Consider the surface $x^2 - e^{xy} + z^2 = 1$.

(a) Find the equation of the tangent plane to the surface at the point $(1, 0, 1)$.

(b) Find the equation of the line perpendicular to the surface, also at $(1, 0, 1)$.

Solution.

(a) Define $F(x, y, z) = x^2 - e^{xy} + z^2$. The tangent plane to the level surface $F = 1$ at the point $(1, 0, 1)$ is given by

$$\begin{aligned} F_x(1,0,1)\cdot(x-1) + F_y(1,0,1)\cdot(y-0) + F_z(1,0,1)\cdot(z-1) &= 0 \\ (2x - ye^{xy})\big|_{(1,0,1)}\cdot(x-1) + (-xe^{xy})\big|_{(1,0,1)}\cdot y + (2z)\big|_{(1,0,1)}\cdot(z-1) &= 0 \\ 2\cdot(x-1) - y + 2\cdot(z-1) &= 0 \\ 2x - y + 2z - 4 &= 0 \end{aligned}$$

(b) The line perpendicular to the surface at $(1, 0, 1)$ is given by

$$l(s) = (1, 0, 1) + s\cdot(2, -1, 2).$$

(3) Is the point $(-1, 0)$ a relative maximum, minimum, saddle point, or none of these for the function

$$f(x, y) = \frac{x^3 - 3x}{1 + y^2}?$$

If it is a relative maximum or minimum is it also the absolute one?

Solution. We calculate the gradient of f first:

$$\nabla f(x, y) = \left(\frac{3x^2 - 3}{1 + y^2}, -\frac{x^3 - 3x}{(1 + y^2)^2}2y\right)$$

So that

$$\nabla f(-1, 0) = (0, 0)$$

so this indeed is a critical point. (Function has one more critical point at $(1, 0)$.) To determine the type of critical point, we compute the matrix of second derivatives:

$$H(f)(x, y) = \begin{pmatrix} \frac{6x}{1+y^2} & -\frac{3x^2-3}{(1+y^2)^2}2y \\ -\frac{3x^2-3}{(1+y^2)^2}2y & -2\frac{x^3-3x}{(1+y^2)^2} + 8y^2\frac{x^3-3x}{(1+y^2)^3} \end{pmatrix}$$

so that

$$H(f)(-1, 0) = \begin{pmatrix} -6 & 0 \\ 0 & -4 \end{pmatrix}$$

which has determinant 24. Since $f_{xx}(-1,0) = -6 < 0$, this is a local maximum. This is not the global maximum since f tends to $+\infty$ when x tends to $+\infty$.

(4) Consider the function

$$f(x,y) = 2x^2y - x^2 - y^2.$$

(a) Find and classify the critical points of f as local maxima, minima, saddles or neither.
(b) Using the first part, can you determine if the *maximum* of f on the region $x^2 + y^2 \leq 1$ must be on the boundary circle $x^2 + y^2 = 1$? Justify! Don't actually find the extreme points along the boundary of the disk.

Solution.

(a) We begin by finding the gradient:

$$\begin{aligned} \nabla f(x,y) &= (4xy - 2x, 2x^2 - 2y) \\ &= 2(2xy - x, x^2 - y). \end{aligned}$$

The critical points can be found by solving the two equations:

$$\begin{aligned} 2xy - x &= 0 \\ x^2 - y &= 0. \end{aligned}$$

Plugging the second into the first we get $2x^3 = x$, so that either $x = 0, y = 0$ or $x = \pm\frac{1}{\sqrt{2}}, y = \frac{1}{2}$.

Now we calculate the matrix of second derivatives:

$$H(f)(x,y) = \begin{pmatrix} 4y-2 & 4x \\ 4x & -2 \end{pmatrix},$$

so that

$$H(f)(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

has determinant 4,

$$H(f)(\frac{1}{\sqrt{2}}, \frac{1}{2}) = \begin{pmatrix} 0 & 4/\sqrt{2} \\ 4/\sqrt{2} & -2 \end{pmatrix},$$

has determinant -8,

$$H(f)(\frac{-1}{\sqrt{2}}, \frac{1}{2}) = \begin{pmatrix} 0 & -4/\sqrt{2} \\ -4/\sqrt{2} & -2 \end{pmatrix},$$

has determinant -8.

Since $f_{xx}(0,0) < 0$, $(0,0)$ is a local maximum. $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$ are saddles.

(b) There is no way for us to tell where the maximum of f on the unit disk $x^2 + y^2 \leq 1$ is, because of the two saddle points. The max could either be $(0,0)$ or somewhere on the boundary. We'd have to inspect the boundary separately.

(5) The height z of a mountain above the point (x,y) is given by

$$z = x(4 - \cos y).$$

(a) Starting at the point $(1, \pi/2)$, what is the direction of steepest *descent*?
(b) If there's a trail on the mountain that lies over the path (t, t^2), what is the slope *along the trail* at $t = \sqrt{\pi}$?

Solution.

(a) Denote $h(x,y) = x(4 - \cos y)$. Let us calculate the gradient of h:

$$\nabla h(x,y) = (4 - \cos y, x \sin y) \Rightarrow \nabla h(1, \pi/2) = (4, 1).$$

And, thus, the direction of steepest descent is the direction $-\frac{(4,1)}{\|(4,1)\|}$ (we usually normalize a vector when merely talking of its *direction*).

(b) *Wrong solution:*
Denoting $c(t) = (t, t^2)$, the elevation of the trail as a function of time is given by

$$g(t) = h(c(t)) = t(4 - \cos t^2)$$

so that $g'(t) = 4 - \cos t^2 + 2t^2 \sin t^2$. Evaluating at $t = \sqrt{\pi}$, we get $g'(t) = 5$.
Alternatively, we could use the chain rule:

$$\begin{aligned}\frac{d}{dt}h(c(t)) &= \nabla h(c(t)) \cdot c'(t) \\ &= (4 - \cos t^2, t \sin t^2) \cdot (1, 2t) \\ &= 4 - \cos t^2 + 2t^2 \sin t^2.\end{aligned}$$

Evaluating at $t = \sqrt{\pi}$ we have $\frac{d}{dt}h(c(\sqrt{\pi})) = 5$.

Correct solution:
What did we do wrong? Everything seems fine, *but* recall that, in fact, the slope is just the directional derivative, and therefore the tangent vector must be normalized.

The speed along the path at time $t = \sqrt{\pi}$ is $\|c'(\sqrt{\pi})\| = \|(1, 2\sqrt{\pi})\| = \sqrt{1 + 4\pi}$. Thus, the correct solution, would be to use the formula for the directional derivative:
We then get

$$\begin{aligned}\text{slope} &= \nabla h(c(t)) \cdot \frac{c'(t)}{\|c'(t)\|} \\ &= (4 - \cos t^2, t \sin t^2) \cdot \frac{(1, 2t)}{\|(1, 2t)\|} \\ &= (4 - \cos t^2, t \sin t^2) \cdot \frac{(1, 2t)}{\sqrt{1 + 4t^2}} \\ &= \frac{4 - \cos t^2 + 2t^2 \sin t^2}{\sqrt{1 + 4t^2}}.\end{aligned}$$

Evaluating at $t = \sqrt{\pi}$ we have that the slope is $5/\sqrt{1 + 4\pi}$.

(6) Let $f(x, y) = (x + 1)^2 + (y - 1)^2 - 4$.
(a) Sketch the surface $z = f(x, y)$.
(b) Sketch the level curves of $z = f(x, y)$ for $z = -4, 0, 5$.
Solution
(a) The surface is a paraboloid shown in Figure 1.
(b) The level curve $f(x, y) = -4$ is just the point $(-1, 1)$ and the other two $f(x, y) = 0$ and 5 are circles centered at that point of radius 2 and 3. See Figure 2.

(7) Let (r, θ, ϕ) be spherical coordinates, i.e.

$$\begin{aligned}x &= \rho \sin \phi \cos \theta, \\ y &= \rho \sin \phi \sin \theta, \\ z &= \rho \cos \phi.\end{aligned}$$

(a) Describe the surfaces $\phi = \pi/4$ and $\rho = \cos \phi$.
(b) Sketch the solid lying above $\phi = \pi/4$ and below $\rho = 1$.
Solution

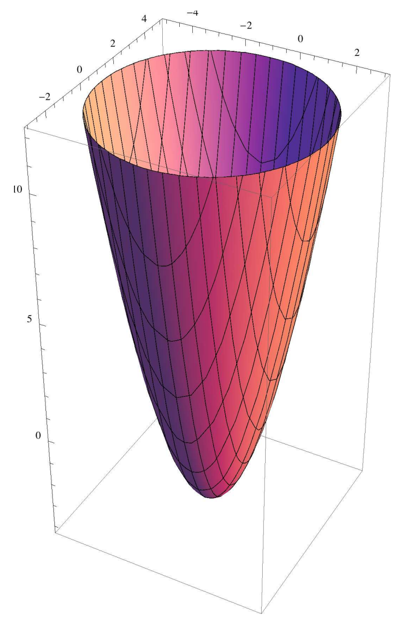

FIGURE 1.

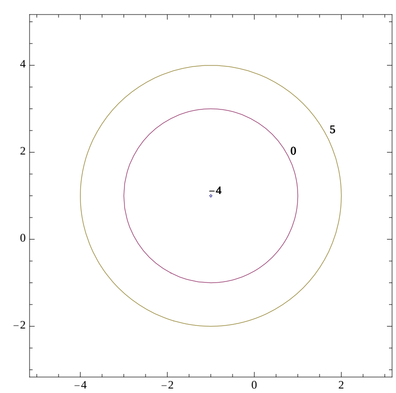

FIGURE 2.

(a) The surface $\phi = \pi/4$ is a cone because ϕ is the angle between the positive z axis and (x, y, z) and for this surface ϕ is constant. In rectangular coordinates the equation of the cone is $z = \sqrt{x^2 + y^2}$.
Since $\rho = \sqrt{x^2 + y^2 + z^2}$ and $\cos\phi = z/\sqrt{x^2 + y^2 + z^2}$, in rectangular coordinates the surface $\rho = \cos\phi$ is $z = x^2 + y^2 + z^2$. This is equivalent to

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

which is the equation of the sphere centered at $(0, 0, 1/2)$ of radius $1/2$.
(b) The solid is an ice cream cone. See Figure 3

(8) (a) Sketch the graph of $\mathbf{r}(t) = (a\cos t, b\sin t, -1)$ for some positive constants a and b.
(b) Find a unit normal vector to this curve at $t = \pi/4$.

Solution

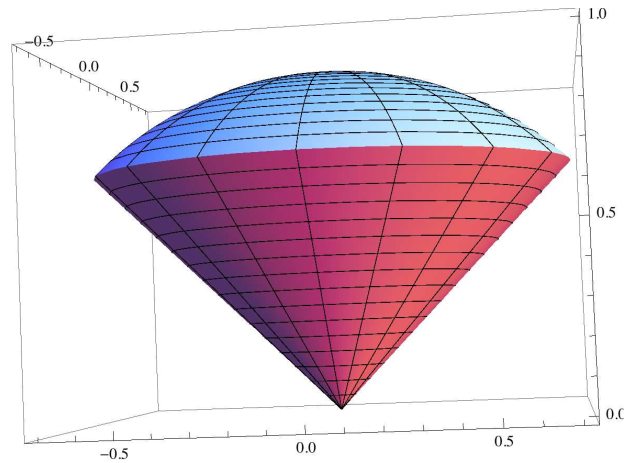

FIGURE 3.

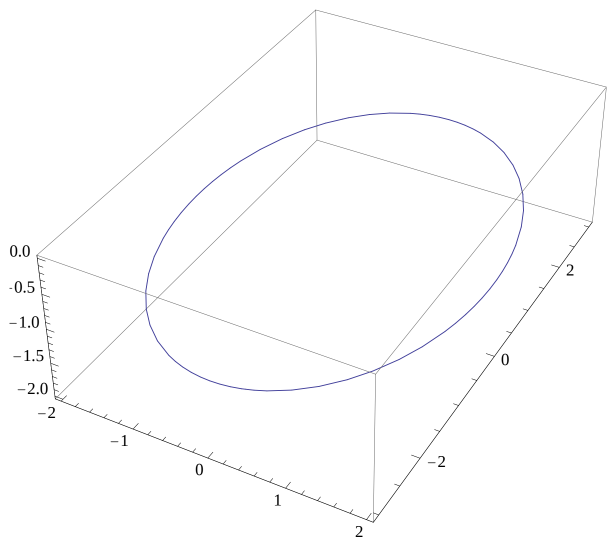

FIGURE 4.

(a) The graph is an ellipse lying in $z = -1$ plane with diameters along x and y directions equal to $2a$ and $2b$. The graph when $a = 2$ and $b = 3$ is shown in Figure 4.

(b) A tangent vector to the ellipse is $\mathbf{r}'(t) = (-a\sin t, b\cos t, 0)$. Since a unit normal vector $\mathbf{N}(t)$ satisfies $\mathbf{N}(t)\cdot\mathbf{r}'(t) = 0$ and $||\mathbf{N}(t)|| = 1$ we can guess that a unit normal vector is

$$\mathbf{N}(t) = \pm\frac{1}{\sqrt{(a\sin t)^2 + (b\cos t)^2}}(b\cos t, a\sin t, 0).$$

At $t = \pi/4$ a unit normal vector is

$$\pm\frac{1}{\sqrt{a^2+b^2}}(b, a, 0).$$

Alternatively, we can compute a normal vector as the derivative of a unit tangent vector. So, we start from a unit tangent vector

$$\mathbf{T}(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-a\sin t, b\cos t, 0).$$

and take the derivative to get

$$\mathbf{n}(t) = \mathbf{T}'(t) = \ldots \text{after some calculation} \cdots = -\frac{ab(b\cos t, a\sin t, 0)}{\sqrt{((a\sin t)^2 + (b\cos t)^2)^3}}.$$

After normalization of $\mathbf{n}(t)$ we get $\mathbf{N}(t)$. The calculation for the derivative of the first component looks:

$$\begin{aligned}\left(\frac{-a\sin t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}\right)' &= \frac{-a\cos t\sqrt{a^2\sin^2 t + b^2\cos^2 t} + \dfrac{a\sin t(a^2\sin t\cos t - b^2\cos t\sin t)}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}}{a^2\sin^2 t + b^2\cos^2 t} \\ &= \frac{-a\cos t(a^2\sin^2 t + b^2\cos^2 t) + a\sin t(a^2\sin t\cos t - b^2\cos t\sin t)}{\sqrt{(a^2\sin^2 t + b^2\cos^2 t)^3}} \\ &= \frac{-ab^2\cos t}{\sqrt{(a^2\sin^2 t + b^2\cos^2 t)^3}}.\end{aligned}$$

(9) Determine if the limit exists and if it exists find its value:

$$\lim_{(x,y)\to(1,-2)} \frac{xy + 2x - y - 2}{(x-1)^2 + (y+2)^2}.$$

Solution Observe that

$$\frac{xy + 2x - y - 2}{(x-1)^2 + (y+2)^2} = \frac{(x-1)(y+2)}{(x-1)^2 + (y+2)^2}$$

and so in the limit both denominator and numerator tend to 0.

The limit does not exist for the same reason as $\lim_{(x,y)\to(0,0)} xy/(x^2+y^2)$ does not exist. We can find two different lines through $(1,-2)$ such that the limits along these lines are different.

Take, for example, $L_1 : x = 1 + t, y = -2$ and $L_2 : x = 1 + t, y = -2 + t$. Then

$$\lim_{t\to 0} \frac{(1+t-1)(-2+2)}{(1+t-1)^2 + (-2+2)^2} = 0$$

and

$$\lim_{t\to 0} \frac{(1+t-1)(-2+t+2)}{(1+t-1)^2 + (-2+t+2)^2} = \frac{1}{2}.$$

(10) The volume of a cylinder of radius r and height h is given by $V = r^2 h\pi$. Suppose that both the height and radius increase from 10 to 10.05 in. Using the linear approximation estimate the increase in volume.

Solution $V_r = 2rh\pi$ and $V_h = r^2\pi$. The local linear approximation of V at $(10, 10)$ is

$$L(r,h) = V(10,10) + V_r(10,10)(r-10) + V_h(10,10)(h-10)$$

and the increase in the volume is

$$V_r(10,10)\cdot 0.05 + V_h(10,10)\cdot 0.05 = (200\cdot 0.05 + 100\cdot 0.05)\pi = 15\pi \approx 47.1 \text{ in}^3.$$