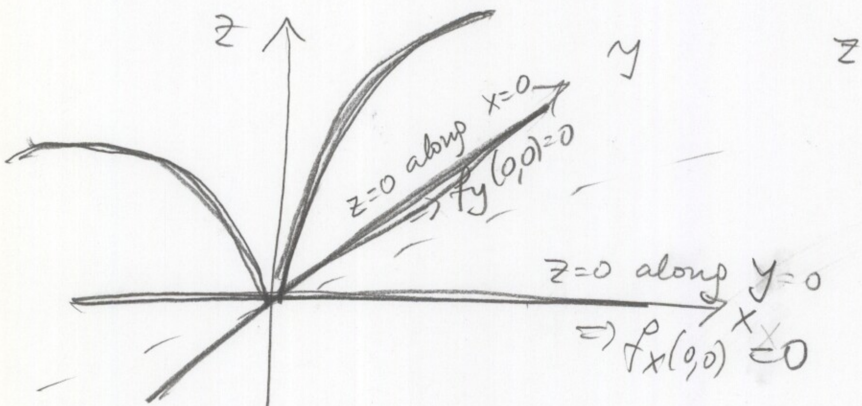
PRACTICE EXAM SOLUTIONS

1. a) TRUE. by Theorem 13.7.1

b) TRUE. by Theorem 13.7.2 $f_x(1,1)=3$, $f_y(1,1)=4$

c) FALSE. f is not differentiable at $(0,0)$. See the figure



$z=f(x,y)$

$D_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)} f(0,0)$ does not exist

no tangent plane at $(0,0)$

2. a) Let $h=g\circ f$ $\quad h:\mathbb{R}^3\to\mathbb{R}$ $\quad g=g(u,v)$ $\quad g_u(1,0)=1$, $g_v(1,0)=-3$

$$h_x = g_u\cdot u_x + g_v\cdot v_x = 1\cdot 0-3\cdot(-3)=9$$

$$h_y = g_u\cdot u_y + g_v\cdot v_y = 1\cdot 0 - 3\cdot(1) = -3$$

$$h_z = g_u\cdot u_z + g_v\cdot v_z\Big|_{(0,0,0)} = 1\cdot 0 - 3\cdot(1) = 3$$

$$u_x\Big|_{(0,0,0)} = -2ye^{-2xy} = 0 \qquad v_x\Big|_{(0,0,0)} = 2x-4+\cos(x+y+z) = -3$$

$$u_y\Big|_{(0,0,0)} = -2xe^{-2xy} = 0 \qquad v_y\Big|_{(0,0,0)} = \cos(x+y+z) = 1$$

$$u_z\Big|_{(0,0,0)} = 0 = 0 \qquad v_z\Big|_{(0,0,0)} = 2z+\cos(x+y+z) = 1$$

$$\nabla h = (9,-3,3)$$

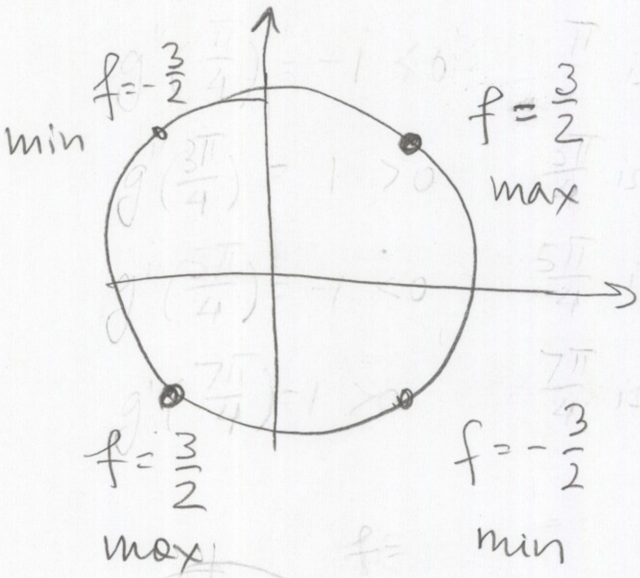
b) $0 = 9x-3y+3z$ or $0=3x-y+z$

3. a) $x = \cos t \qquad y = \sin t \qquad t \in [0, 2\pi)$

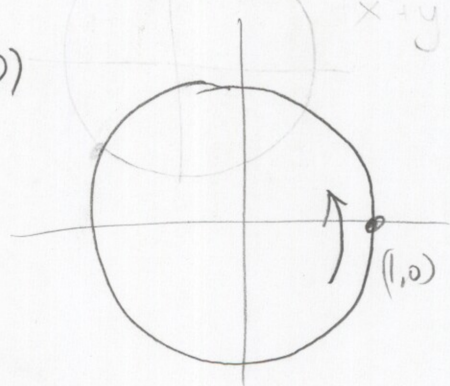
$$f(x,y) = 1 + \cos t \sin t = 1 + \frac{\sin 2t}{2} = g(t)$$

min for $\sin 2t = -1 \qquad t = \frac{3\pi}{4}, \frac{7\pi}{4}$

max for $\sin 2t = 1 \qquad t = \frac{\pi}{4}, \frac{5\pi}{4}$



b)



$1 + \frac{\sin 2t}{2}$ increases

4. $f(x,y) = 5x^2 - 2y^2 + 10$ on $x^2+y^2 \le 1$

f is continuous and $x^2+y^2 \le 1$ is closed and bounded

so abs max and abs min exist

relative min & max

$$f_x = 10x = 0 \qquad (x,y) = (0,0)$$
$$f_y = -2y = 0$$

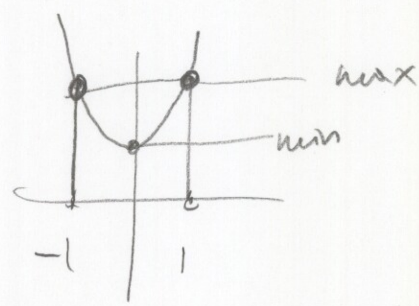
$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 10 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det H < 0 \Rightarrow (0,0)$ is a saddle point

So, max and min are attained on $x^2+y^2=1$.

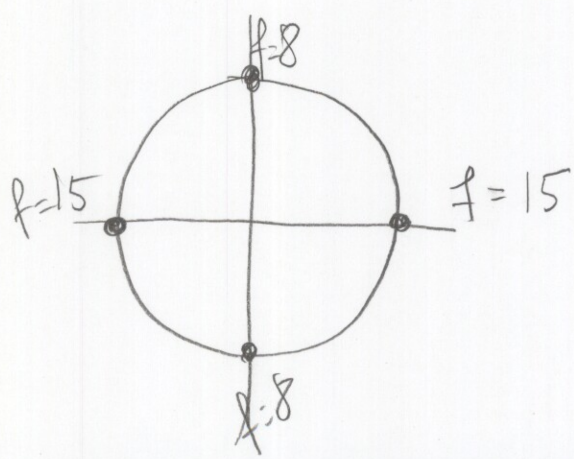
There

$$f(x,y) = 5x^2 - 2 + 2x^2 + 10$$
$$= 7x^2 + 8$$



So min is attained at $(x,y) = (0,1)$ or $(0,-1)$

max $(x,y) = (1,0)$ or $(-1,0)$



5. • Maximum rate of change is in the direction of the gradient at $P(2,-1,0)$:

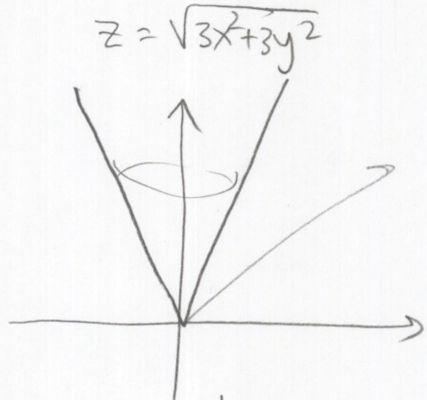
$$\nabla f(2,-1,0) = (2x, 8y, 18z)\big|_{(2,-1,0)} = (4,-8,0)$$

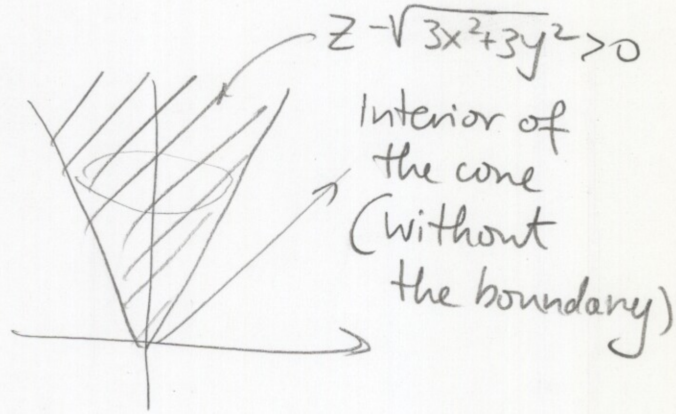
• Equation of the tangent plane at $P(\frac{1}{2}, \frac{1}{3}, -\frac{\sqrt{11}}{8}$ is)

$$0 = f_x\left(\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{\sqrt{11}}{18}\right)\left(x-\tfrac{1}{2}\right) + f_y\left(\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{\sqrt{11}}{8}\right)\left(y-\tfrac{1}{3}\right) + f_z\left(\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{\sqrt{11}}{8}\right)\left(z+\tfrac{\sqrt{11}}{8}\right)$$

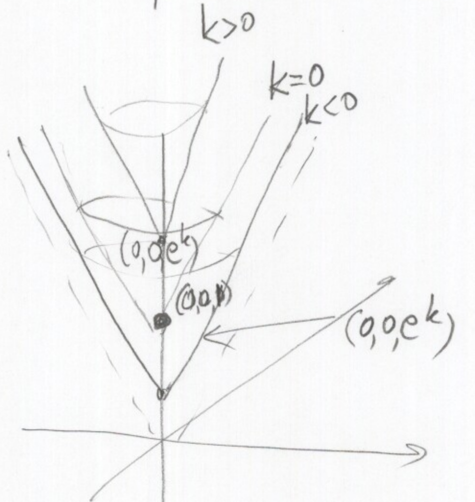
$$= \left(x-\tfrac{1}{2}\right) + \tfrac{8}{3}\left(y-\tfrac{1}{3}\right) - 18\cdot\tfrac{\sqrt{11}}{8}\cdot\left(z+\tfrac{\sqrt{11}}{8}\right)$$

6. a) $D = \{(x,y,z) : z - \sqrt{3x^2+3y^2} > 0\}$





b)



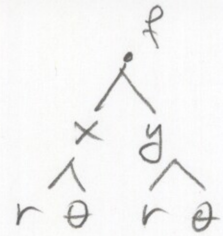
$f = k$ is a cone with the vertex at e^k

for $k<0$ $\quad e^k < 1$

for $k=0$ $\quad e^k = 1$

for $k>0$ $\quad e^k > 1$

7. $f = f(x, y)$



$f = f(x(r, \theta), y(r, \theta))$

$f_r = f_x \cdot x_r + f_y \cdot y_r$

$f_\theta = f_x \cdot x_\theta + f_y \cdot y_\theta$

$x = r\cos\theta$ $\qquad x_r = \cos\theta \qquad y_r = \sin\theta$

$y = r\sin\theta$ $\qquad x_\theta = -r\sin\theta \qquad y_\theta = r\cos\theta$

$f_r = f_x \cdot \cos\theta + f_y \sin\theta \qquad / \cdot r\cos\theta$

$f_\theta = f_x \cdot (-r\sin\theta) + f_y \cdot r\cos\theta \qquad / -\sin\theta$

$f_r \cdot r\cos\theta - f_\theta \sin\theta = f_x \cdot r$

$$\boxed{f_x = f_r \cos\theta - f_\theta \frac{\sin\theta}{r}}$$

$f_r = f_x \cos\theta + f_y \sin\theta \qquad / \cdot r\sin\theta$

$f_\theta = f_x(-r\sin\theta) \;\; f_y r\cos\theta \qquad / \cdot \cos\theta$

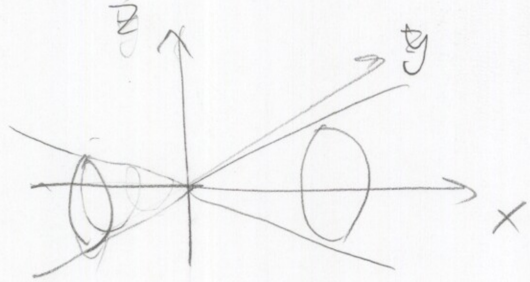
$f_r r\sin\theta + f_\theta \cos\theta = f_y \cdot r$

$$\boxed{f_y = f_r \sin\theta + f_\theta \frac{\cos\theta}{r}}$$

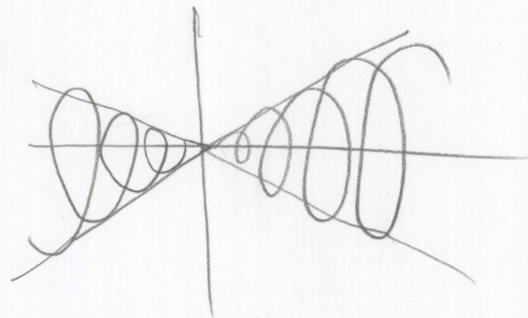
8. a) $x = -t$

$y = t\sin\pi t \quad \Rightarrow \quad x^2 = y^2 + z^2$

$z = t\cos\pi t \qquad\qquad x = \pm\sqrt{y^2+z^2}$ cone



b)



c) $x' = -1$

$y' = \sin\pi t + t\cdot\cos\pi\cdot t\cdot\pi$

$z' = \cos\pi t - t\cdot\sin\pi t\cdot\pi$

$x(1) = -1 \qquad x'(1) = -1$

$y(1) = 0 \qquad y'(1) = -\pi$

$z(1) = -1 \qquad z'(1) = -1$

$x = -1 - t$

$y = 0 - \pi t$

$z = -1 - t$

9. Observe

$$\lim_{t\to 0} \frac{\cos t - 1}{t^2} = \lim_{t\to 0} \frac{-\sin t}{2t} = \lim_{t\to 0} \frac{-\cos t}{2} = -\frac{1}{2}$$

Then if $f(t) = \frac{\cos t - 1}{t^2}$ $\quad g(x,y) = x+y$

$$\frac{\cos(x+y)-1}{(x+y)^2} = f(g(x,y)) \quad \text{and} \quad g(0,0)=0$$

and by the composition rule

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x+y)-1}{(x+y)^2} = -\frac{1}{2}.$$

10.
a) $f_x = \frac{2}{y+z}$ $\quad = 1$

$f_y = \frac{3\cdot(y+z)-(2x+3y)\cdot 1}{(y+z)^2} = \frac{3z-2x}{(y+z)^2}$ $\quad = \frac{5}{4}$

$f_z = -\frac{2x+3y}{(y+z)^2}$ $\quad = -\frac{1}{4}$

$(1,1,1)$

The local linear approximation at $(-1,1,1)$ is

$$L(x,y,z) = \frac{1}{2} + 1\cdot(x+1) + \frac{5}{4}(y-1) - \frac{1}{4}(z-1)$$

b) distance between A&B is $\sqrt{(0.01)^2+(0.01)^2+(0.01)^2} = \frac{\sqrt{3}}{100}$.

error is

$$\left|1\cdot 0{,}01 - \frac{5}{4}\cdot 0{,}01 - \frac{1}{4}\cdot 0.01\right| = \frac{1}{2\cdot 100}$$

Error is actually
1/2*0.99-(1/2-1/2*0.01)=0.
So, no error here.

$$\frac{\text{error}}{\text{dist}} = \frac{\frac{1}{2}}{\sqrt{3}} = \frac{1}{2\cdot\sqrt{3}}$$