1.

a) $F(x,y,z,\lambda) = \overbrace{\log x + \log y + 2\log z}^{f(x,y,z)} - \lambda(x^2+y^2+z^2-4r^2)$

$F_x = \frac{1}{x} - 2\lambda x = 0 \Rightarrow x^2 = \frac{1}{2\lambda}$

$F_y = \frac{1}{y} - 2\lambda y = 0 \Rightarrow y^2 = \frac{1}{2\lambda} \Rightarrow$ $\boxed{\begin{array}{l} x = r \\ y = r \\ z = \sqrt{2}r \end{array}}$ $\lambda = \frac{1}{2r}$

$F_z = \frac{2}{z} - 2\lambda z = 0 \Rightarrow z^2 = \frac{2}{2\lambda}$

extremal point

$$\begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2}-2\lambda & 0 & 0 \\ 0 & -\frac{1}{y^2}-2\lambda & 0 \\ 0 & 0 & -\frac{1}{z^2}-2\lambda \end{bmatrix}$$

Principal minors are negative so $(x,y,z)=(r,r,\sqrt{2}r)$ is relative max of f for the given constraint. It is also the global constrained max since there is only one extremal point. So,

$$\log x + \log y + 2\log z \leq \log r + \log r + 2\log\sqrt{2}r$$

$$\log xyz^2 \leq \log 2r^4$$

$$xyz^2 \leq 2r^4 \qquad (*)$$

b) Let $x = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $y = \frac{b}{\sqrt{a^2+b^2+c^2}}$, $z = \frac{c}{\sqrt{a^2+b^2+c^2}}$ then

$x^2+y^2+z^2 = 1$ i.e. $r = \frac{1}{2}$.

By $(*)$

$$\frac{abc^2}{(a^2+b^2+c^2)^2} \leq 2 \cdot \frac{1}{2^4} = \frac{1}{8}.$$

2. a) f is continuous at every point different than $(0,0)$ since x^3-y^3 and x^2+y^2 are continuous and $x^2+y^2 \neq 0$.

At $(x,y)=(0,0)$ f is continuous by the squeezing theorem since

$$0 \le \left|\frac{x^3-y^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| \le |x|+|y| \to 0$$

b)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3-0^3}{h^2+0^2} - 0}{h} = 1$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0^3-h^3}{0^2+h^2} - 0}{h} = -1$$

c)

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$= 0 + 1(x-0) - 1\cdot(y-0)$$

$$= x-y$$

3. a) $g_u = f_x \cdot x_u + f_y \cdot y_u = f_x \cdot v + f_y \cdot u$

$g_v = f_x \cdot x_v + f_y \cdot y_v = f_x \cdot u - f_y\, v$

$$g_{uv} = (f_x v + f_y \cdot u)_v = (f_{xx} \cdot x_v + f_{xy}\, y_v) \cdot v + f_x$$

$$+ (f_{yx} \cdot x_v + f_{yy} \cdot y_v) \cdot u$$

$$= f_{xx}\, uv - f_{xy}\, v^2 + f_x + f_{yx}\, u^2 - f_{yy}\, uv$$

b)

$$a\, g_u^2 - b\, g_v^2 = a(f_x v + f_y \cdot u)^2 - b(f_x u - f_y v)^2$$

$$= a(f_x^2 v^2 + 2 f_x f_y uv + f_y^2 u^2)$$

$$- b(f_x^2 u^2 - 2 f_x f_y uv + f_y^2 v^2)$$

$$= (a f_y^2 - b f_x^2) u^2 + 2 f_x f_y (a+b) uv + (a f_x^2 - b f_y^2) v^2$$

$$= u^2 + v^2$$

$$\Rightarrow a f_y^2 - b f_x^2 = 1$$

$$a f_x^2 - b f_y^2 = 1$$

$\|\nabla f\|^2$

$\|$

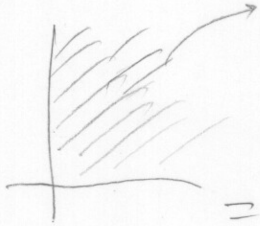
$\Rightarrow a(f_y^2 + f_x^2) = 1$

$a(f_x^2 + f_y^2)^2 = 1$

$$a + b = 0 \quad \Rightarrow a = -b$$

$$\Rightarrow a \cdot 2 = 1 \quad \Rightarrow a = \frac{1}{2} \quad b = -\frac{1}{2}$$

4. a) $$\int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^{+\infty} e^{-r^2} r\,dr\,d\theta = \int_0^{\pi/2} -\frac{1}{2}e^{-r^2}\Big|_0^{+\infty} d\theta$$

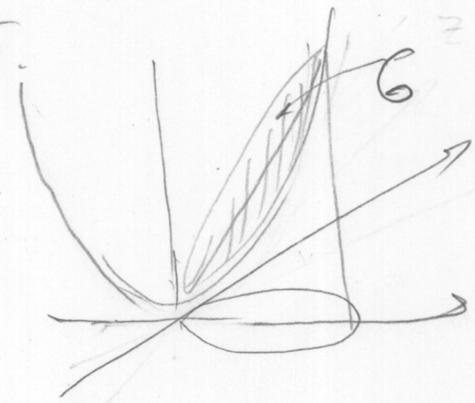


$$= \int_0^{\pi/2} \frac{1}{2}\,d\theta = \frac{\pi}{4}$$

b) $$\int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{-\infty} e^{-x^2}dx \int_0^{-\infty} e^{-y^2}dy = \left(\int_0^{-\infty} e^{-x^2}dx\right)^2 = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{-\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

5.



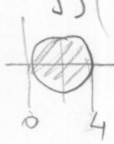
Intersection of

$\underbrace{x^2+y^2=z}_{\text{paraboloid}}$ and $\underbrace{z=4x}_{\text{plane}}$

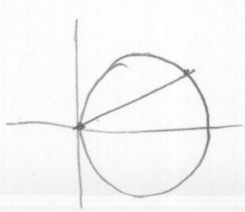
$$x^2+y^2=4x \Rightarrow x^2-4x+4+y^2=4$$
$$\underbrace{(x-2)^2+y^2=4}_{\text{circle in xy plane}}$$

$$I = \text{Volume}(G) = \iiint_G dx\,dy\,dz$$

a) $$I = \iint\left(\int_{x^2+y^2}^{4x} dz\right)dx\,dy = \int_0^4 \int_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} \int_{x^2+y^2}^{4x} dz\,dy\,dx$$



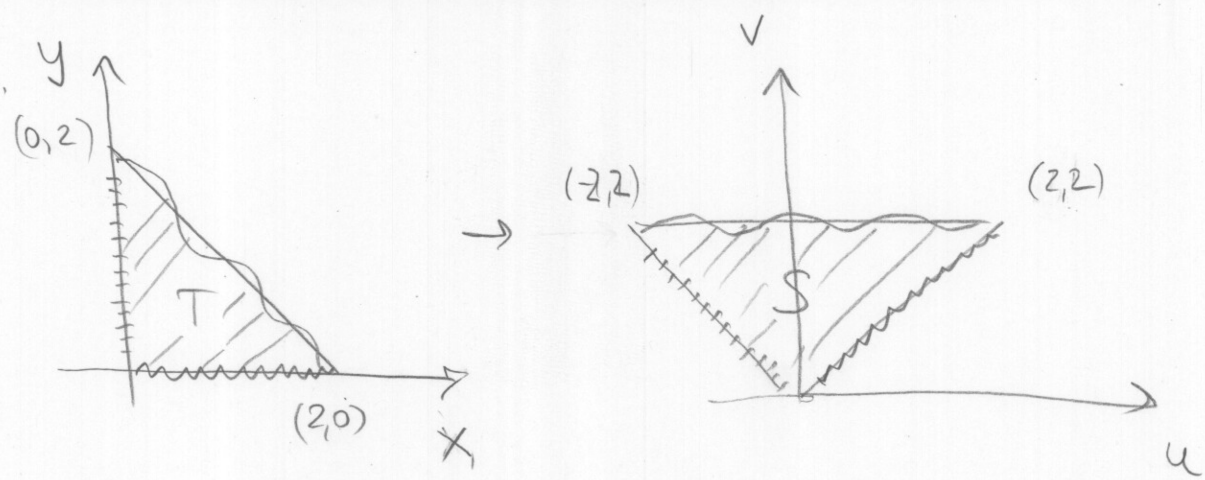
b) $$I = \iint \int_{r^2}^{4r\cos\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{4\cos\theta} \int_{r^2}^{4r\cos\theta} r\,dr\,d\theta$$



$$(r\cos\theta-2)^2+(r\sin t)^2=4$$

$$r^2-4r\cos\theta+\cancel{4}=\cancel{4} \quad r=4\cos\theta$$

6. a)



$x=0 \Rightarrow u=y\,,\ v=y \Rightarrow u=v$

$y=0 \Rightarrow u=-x\,,\ v=x \Rightarrow u=-v$

$x+y=2 \Rightarrow v=2$

$$\boxed{x=\frac{v-u}{2} \qquad y=\frac{u+v}{2}}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

b)

$$\iint_T e^{\frac{y-x}{y+x}}\,dx\,dy = \iint_S e^{\frac{u}{v}}\cdot\frac{1}{2}\,du\,dv = \int_0^2\int_{-v}^{v}\frac{1}{2}e^{\frac{u}{v}}\,du\,dv$$

$$= \int_0^2 \frac{1}{2}\,v\cdot e^{\frac{u}{v}}\Big|_{-v}^{v}\,dv = \int_0^2 \frac{1}{2}\,v(e-e^{-1})\,dv = (e-e^{-1})\cdot\frac{1}{4}v^2\Big|_0^2$$

$$= e-e^{-1}$$