MA 18 FINAL PRACTICE EXAM

Collaboration or use of online help is *forbidden* on the whole practice test until Tuesday at noon. After that you can discuss it with your peers, TAs and instructors.

You can turn in the five problems marked with $\star$ for *extra credit*. You should turn them in to your TA in recitations on Tuesday, December 7 at noon. If you can't make it to the recitations put them into the section box in the Kassar House before Tuesday at noon. Other than the due time there is no time limit on how many hours you can spend on these problems. Write solutions nicely, explain your reasoning and make sure your handwriting is readable.

Completely correct solutions on these five problems can bring you extra 10 points on your final. The final will be worth 100 points.

(1) (a) Find and classify extremal points of $\log x + \log y + 2\log z$ on the portion of the sphere $x^2+y^2+z^2 = 4r^2$ where $x > 0$, $y > 0$, $z > 0$.
  (b) Use the result to prove that for real positive numbers a, b, c we have
$$abc^2 \le \frac{(a^2+b^2+c^2)^2}{8}.$$

(2)$\star$ Let
$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$
  (a) Show that f is continuous at every point.
  (b) Find $f_x(0,0)$ and $f_y(0,0)$. *Hint:* by definition.
  (c) Find the local linear approximation to f at (0,0).

(3) The change of variables $x = uv$ and $y = \frac{1}{2}(u^2-v^2)$ transforms $f(x,y)$ to $g(u,v)$.
  (a) Calculate $\partial g/\partial u$, $\partial g/\partial v$ and $\partial^2 g/\partial u \partial v$ in terms of partial derivatives of f. (You may assume equality of mixed partials.)
  (b) If $||\nabla f(x,y)||^2 = 2$ for all x and y, determine constants a and b such that
$$a\left(\frac{\partial g}{\partial u}\right)^2 - b\left(\frac{\partial g}{\partial v}\right)^2 = u^2+v^2.$$

(4) (a) Evaluate the integral by converting it to polar coordinates:
$$\int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}dxdy.$$
  (b) Use the result in part (a) to compute
$$\int_0^{+\infty} e^{-x^2}dx.$$

(5) Let $G = \{(x,y,z) : x^2+y^2 \le z \le 4x\}$. Express the volume of G as an iterated integral in
  (a) rectangular coordinates,
  (b) cylindrical coordinates.

(6)$\star$ Let T be the triangle bounded by the line $x+y=2$ and the two coordinate axes. A change of variables is given by the transformation
$$u = y - x, \quad v = y + x.$$
  (a) Find the image of T under this transformation and compute the Jacobian determinant
$$\frac{\partial(x,y)}{\partial(u,v)}.$$
  (b) Use the transformation to compute
$$\iint_T e^{(y-x)/(y+x)}dxdy.$$

(7) ⋆ Let W be the three-dimensional region under the graph of $f(x,y) = \exp(x^2+y^2)$ and over the region in the plane defined by $1 \le x^2+y^2 \le 2$.
  (a) Find the volume of W.
  (b) Find the flux of the vector field $\mathbf{F} = (2x - xy)\mathbf{i} - y\mathbf{j} + yz\mathbf{k}$ out of the boundary ∂W.

(8) Let $\mathbf{F}(x,y) = 2xy\mathbf{i} + x^2\mathbf{j}$. Show that the integral of $\mathbf{F}$ around the circumference of the square $[0,1]\times[0,1]$ in the xy plane is 0 by
  (a) direct evaluation.
  (b) using Green's theorem.

(9) ⋆ Let

$$\mathbf{F}(x,y,z) = e^{xy}\left[(yz + xy^2z)\mathbf{i} + (xz + x^2yz)\mathbf{j} + xy\mathbf{k}\right].$$

  (a) Show that $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^3 \to \mathbb{R}$ (there's no need to actually find f).
  (b) Let C be the curve obtained by intersecting the sphere $x^2+y^2+z^2 = 1$ with the plane $x = 1/2$, and let S be the portion of the sphere with $x \ge 1/2$. Draw a figure including possible (compatible) orientations for C and S. State Stokes' theorem for this region.
  (c) With $\mathbf{F}$ as in (a), let $\mathbf{G} = \mathbf{F} + (z-y)\mathbf{i} + y\mathbf{k}$. Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}$$

  with your preferred orientation, where S is as in (b). Here, $\nabla\times\mathbf{G} = \operatorname{curl}\mathbf{G}$ and $d\mathbf{S} = \mathbf{n}\,dS$, where $\mathbf{n}$ is a unit normal to the surface. *Hint:* $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$.
  (d) Can you find f in (a)?

(10) (a) Let S be the surface $x^2 + 2y^2 + 2z^2 = 1$. Find a parametrization of S and use it to find the tangent plane to S at $(1/\sqrt{2}, 1/2, 0)$.
  (b) Verify that the curve $\mathbf{c}(t) = \cos t\ \mathbf{i} + (1/\sqrt{2})\sin t\ \mathbf{j}, 0 \le t \le 2\pi$ lies in the surface S, and that $\mathbf{c}'(\pi/4)$ lies in the tangent plane found in (a).
  (c) Write down an integral representing the area of the surface using the parametrization you found.

(11) ⋆ Answer the following short questions: If true, justify, if false give a counterexample.
  (a) The path integral $\int_{\mathbf{c}} 2\pi\, ds$ is the surface area of a cylinder of radius 1 and height 2π where the path is defined by $\mathbf{c} = (\cos t, \sin t, 0)$, and $0 \le t \le 2\pi$.
  (b) There is no vector field $\mathbf{F}$ such that $\nabla\times\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(12) Let C be the circle $x^2+y^2 = 1, z = 0$ and let

$$\mathbf{F}(x,y,z) = [x^2y^3 + y - \cos(x^3)]\mathbf{i} + [x^3y^2 + \sin(y^3) + x]\mathbf{j} + z\mathbf{k}$$

Calculate the line integral $\int_C \mathbf{F}\cdot d\mathbf{s}$.