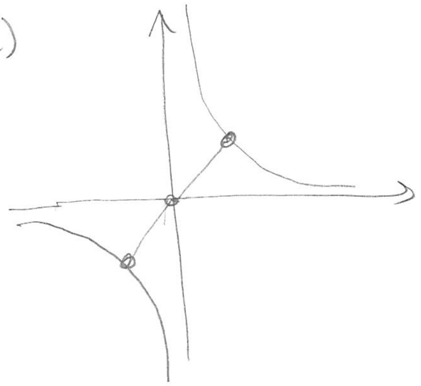
1. a)



From the graph we see that the distance is minimal at (1,1) and (-1,-1).

b) The distance is $\sqrt{x^2+y^2}$. We will minimize $f(x,y)=x^2+y^2$ instead of $\sqrt{x^2+y^2}$.

$F(x,y,\lambda)=x^2+y^2-\lambda(xy-1)$

$F_x=2x-\lambda y=0 \quad \Rightarrow \quad \lambda=\frac{2x}{y}$

$F_y=2y-\lambda x=0 \quad\quad\quad \lambda=\frac{2y}{x}$

$\left(\begin{array}{c} x\neq 0 \,\&\, y\neq 0 \text{ for} \\ xy=1 \end{array}\right) \Bigg\} \Rightarrow x^2=y^2$

$F_\lambda=xy-1=0$

So, $xy=1$ and $x^2=y^2 \Rightarrow (x,y)=(1,1)$ or $(-1,-1)$

min distance is 1. max is not attained

c) $y=\frac{1}{x}$ $\quad f(x,\frac{1}{x})=x^2+\frac{1}{x^2}$

$g(x)=x^2+\frac{1}{x^2}$

$g'(x)=2x-2\frac{1}{x^3}=2\frac{x^4-1}{x^3}=0 \quad x=\pm 1$

$g''(x)=2+\frac{6}{x^4}>0$ min

2. The derivative is maximal in the direction of the gradient and is equal to the norm of the gradient.

$$\nabla f = (ay^2 + 3cz^2x^2,\ 2axy + bz,\ by + 2czx^3)$$

$$\nabla f(1,2,1) = (4a+3c,\ 4a+b,\ 2b+2c) = (0,0,64)$$

$$\left.\begin{aligned} 4a+3c &= 0 \\ 4a+b &= 0 \\ 2b+2c &= 64 \end{aligned}\right\} \quad \begin{aligned} a &= -6 \\ b &= 24 \\ c &= 8 \end{aligned}$$

3. a) $\vec{N} = (-z_x, -z_y, 1)$

$$\vec{N} = \left( -\frac{x}{\sqrt{x^2+y^2}} - 3x\sqrt{x^2+y^2},\ -\frac{y}{\sqrt{x^2+y^2}} - 3y\sqrt{x^2+y^2},\ 1 \right)$$

$$= \frac{1}{\sqrt{x^2+y^2}} \left( -x(1+3(x^2+y^2)),\ -y(1+3(x^2+y^2)),\ \sqrt{x^2+y^2} \right)$$

b) $\cos\theta = \dfrac{\vec{N}\cdot\vec{k}}{\|\vec{N}\|} = \dfrac{1}{\|\vec{N}\|}$

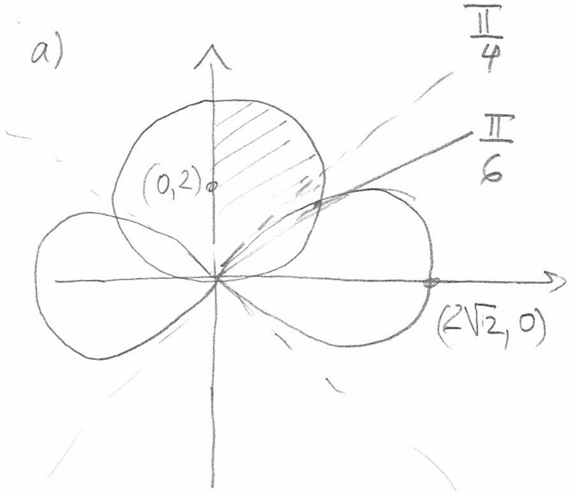
$$\|\vec{N}\| = \frac{1}{\sqrt{x^2+y^2}} \sqrt{x^2(1+3(x^2+y^2))^2 + y^2(1+3(x^2+y^2))^2 + x^2+y^2}$$

$$= \frac{1}{\sqrt{x^2+y^2}} \sqrt{(x^2+y^2)\left((1+3(x^2+y^2))^2+1\right)} = \sqrt{(1+3(x^2+y^2))^2+1}$$

$$\cos\theta = \frac{1}{\sqrt{(1+3(x^2+y^2))^2+1}}$$

$$\lim_{(x,y,z)\to(0,0,0)} \cos\theta = \frac{1}{\sqrt{2}}$$

4. a)




$r = 4\sin\theta$
$r^2 = 8\cos 2\theta$ $\}$ $\sin\theta = \pm\frac{1}{2}$

$\theta = \pm\frac{\pi}{6}$

b)
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{\sqrt{8\cos 2\theta}}^{4\sin\theta} r\,dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{4\sin\theta} r\,dr\,d\theta =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \Big|_{\sqrt{8\cos 2\theta}}^{4\sin\theta} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_{0}^{4\sin\theta}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}\left[(4\sin\theta)^2 - 8\cos 2\theta\right] d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\, 16\sin^2\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}\left[8 - 8\cos 2\theta - 8\cos 2\theta\right] d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (4 - 4\cos 2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (4 - 8\cos 2\theta)\, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (4 - 4\cos 2\theta)\, d\theta$$

$$= 4\theta - 4\sin 2\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} + 4\theta - 2\sin 2\theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 2\pi - \frac{2\pi}{3} - 4\left(1 - \frac{\sqrt{3}}{2}\right) - 2(0-1) = \frac{4\pi}{3} - 2 + 2\sqrt{3}$$

5.

$$I_x = \iiint_V x(x^2+y^2+z^2)\,dV = \int_0^1\int_0^1\int_0^1 x^3+xy^2+xz^2\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{2}x^2z^2\Big|_0^1\,dy\,dz = \int_0^1\int_0^1 \left(\frac{1}{4}+\frac{1}{2}y^2+\frac{1}{2}z^2\right)dy\,dz$$

$$= \int_0^1 \frac{1}{4}y+\frac{1}{6}y^3+\frac{1}{2}yz^2\Big|_0^1\,dz = \int_0^1 \frac{1}{4}+\frac{1}{6}+\frac{1}{2}z^2\,dz$$

$$= \frac{1}{4}+\frac{1}{6}+\frac{1}{6} = \frac{7}{12}$$

$$M = \iiint_V dV = 1$$

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{7}{12},\frac{7}{12},\frac{7}{12}\right)$$

6. $u = xy$

$v = yz \qquad x = \sqrt{\frac{uw}{v}}, \quad y = \sqrt{\frac{uv}{w}}, \quad z = \sqrt{\frac{vw}{u}}$

$w = xz$

$$J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{w}{uv}} & -\frac{1}{2}\sqrt{\frac{uw}{v^3}} & \frac{1}{2}\sqrt{\frac{u}{vw}} \\ \frac{1}{2}\sqrt{\frac{v}{uw}} & \frac{1}{2}\sqrt{\frac{u}{vw}} & -\frac{1}{2}\sqrt{\frac{uv}{w^3}} \\ -\frac{1}{2}\sqrt{\frac{vw}{u^3}} & \frac{1}{2}\sqrt{\frac{w}{uv}} & \frac{1}{2}\sqrt{\frac{v}{uw}} \end{vmatrix}$$

$$= \frac{1}{8}\left[ \cancel{\frac{1}{\sqrt{uvw}}} - \cancel{\frac{1}{\sqrt{uvw}}} + \frac{1}{\sqrt{uvw}} + \frac{1}{\sqrt{uvw}} + \frac{1}{\sqrt{uvw}} + \frac{1}{\sqrt{uvw}} \right]$$

$$= \frac{1}{2} \frac{1}{\sqrt{uvw}}$$

$$\text{Volume} = \frac{1}{2} \int_1^2 \frac{1}{\sqrt{u}}\,du \int_1^3 \frac{1}{\sqrt{v}}\,dv \int_1^4 \frac{1}{\sqrt{w}}\,dw =$$

$$= \frac{1}{2}\, 2\sqrt{u}\Big|_1^2 \; 2\sqrt{v}\Big|_1^3 \; 2\sqrt{w}\Big|_1^4$$

$$= 4(\sqrt{2}-1)(\sqrt{3}-1)$$

(7) In 3-space, let $\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $r(x,y,z) = ||\mathbf{r}(x,y,z)||$.

(a) Show that $\nabla r(x,y,z)$ is a unit vector in the direction of $\mathbf{r}(x,y,z)$.

$$r(x,y,z) = \|\vec{r}(x,y,z)\| = \|x\hat{i} + y\hat{j} + z\hat{k}\| = \sqrt{x^2+y^2+z^2}$$

$$\nabla r(x,y,z) = (r_x, r_y, r_z)$$

$$r_x = \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similarly $r_y = \frac{y}{r}$, $r_z = \frac{z}{r}$

$$\therefore \quad \nabla r(x,y,z) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{1}{r}(x,y,z) = \frac{\vec{r}}{r}$$ which is parallel to $\vec{r}$.

Moreover: $\|\nabla r\| = \left\|\frac{\vec{r}}{r}\right\| = \frac{1}{r}\|\vec{r}\| = \frac{1}{r}\cdot r = 1$

(b) Show that $\nabla(r^n) = nr^{n-2}\mathbf{r}$ if $n$ is a positive integer. *Hint*: What is $\nabla(fg)$?

Method #1: Induction. We show for $n=1$, assume for general $n$, and show for $n+1$.

$n=1$: $\nabla(r^1) = \nabla r = \frac{\vec{r}}{r} = 1\cdot r^{1-2}\vec{r}$ by (a)

$n+1$: $\nabla(r^{n+1}) = \nabla(r\cdot r^n) = (\nabla r)r^n + r(\nabla r^n)$

induction assumption

$$= \frac{\vec{r}}{r}\cdot r^n + r\cdot n r^{n-2}\vec{r}$$

$$= r^{n-1}\vec{r} + n r^{n-1}\vec{r} = (n+1)r^{n-1}\vec{r}$$

Method #2: Direct calculation: $\nabla r^n = \left((r^n)_x, (r^n)_y, (r^n)_z\right)$

For example, we calculate $(r^n)_x$:

$$(r^n)_x = \frac{\partial}{\partial x}(r^n) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} = \frac{n}{2}\cdot 2x\cdot(x^2+y^2+z^2)^{\frac{n-2}{2}} = n\cdot r^{n-2}x$$

Repeating the same calculations for $(r^n)_y$, $(r^n)_z$ we get the result.

**This problem is continued on the next page...**

(c) Is the formula in part (b) valid when $n$ is a negative integer or zero? Yes.

$\underline{n=0:}$ Can see by direct evaluation.

$\underline{n<0:}$ We can either see this by direct evaluation (Method 2 works).

or, here's a trick:

$$\vec{0} = \nabla(1) = \nabla(r^n \cdot r^{-n}) = (\nabla r^n)\, r^{-n} + r^n (\nabla r^{-n})$$

apply part (b) to $-n$ which is positive

$$= (\nabla r^n)\, r^{-n} + r^n (-n)\, r^{-n-2}\, \vec{r}$$

$$\Longrightarrow (\nabla r^n)\, r^{-n} = -r^n (-n)\, r^{-n-2}\, \vec{r} = n r^{-2}\, \vec{r}$$

$$\Longrightarrow (\nabla r^n) = n r^{n-2}\, \vec{r}$$

(d) Find a scalar field $f$ such that $\nabla f = \mathbf{r}$.

We seek $f(x,y,z)$ that satisfies $(f_x, f_y, f_z) = (x,y,z)$

It is simple to see that $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$

would satisfy this.

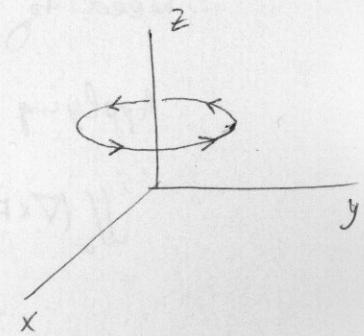
(8) Let $C$ be the curve $x^2 + y^2 = 1$ lying in the plane $z = 1$. Let $\mathbf{F}(x, y, z) = (z - y)\mathbf{i} + y\mathbf{k}$.

(a) Choose and specify an orientation for $C$, sketch in detail, and calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly (don't use any theorem).

Parametrize: $\vec{r}(t) = (\cos t, \sin t, 1) \quad 0 \le t \le 2\pi.$

$\vec{r}\,'(t) = (-\sin t, \cos t, 0)$

$\vec{F}(\vec{r}(t)) = (1 - \sin t, 0, \sin t)$

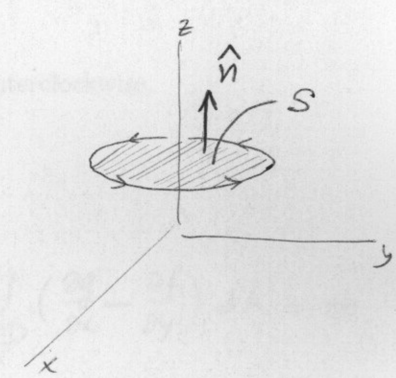



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (1 - \sin t, 0, \sin t) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} (-\sin t + \sin^2 t)\, dt = \pi$$

(b) Choose a convenient surface $S$ that has $C$ as its oriented boundary, sketch in detail, and verify Stokes' theorem (that is, calculate the other integral appearing in the theorem, and verify your answer in (a)).

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$



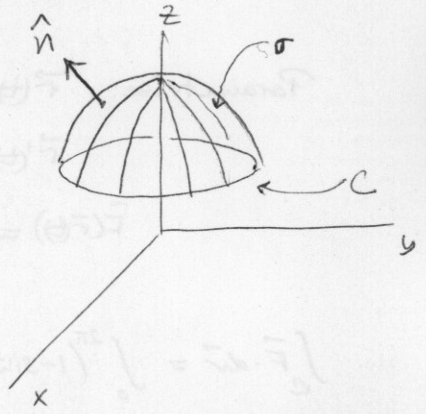

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z-y & 0 & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\hat{n} = (0, 0, 1) \quad \Rightarrow \quad (\nabla \times \vec{F}) \cdot \hat{n} = 1$$

$$\Rightarrow \quad \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \iint_S dS = \text{surface area of } S = \pi$$

**This problem is continued on the next page...**

(c) Consider the sphere with radius $\sqrt{2}$ and center at the origin. Let $\sigma$ be the part of the sphere that is above the plane $z = 1$. Evaluate the flux of $\nabla \times \mathbf{F}$ through $\sigma$. Specify the orientation you are using for $\sigma$.




Need to calculate: $\iint_\sigma (\nabla \times \vec{F}) \cdot \hat{n}\, dS$

Applying Stokes' Theorem, we get

$$\iint_\sigma (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \int_C \vec{F} \cdot d\vec{r} = \pi$$

by part (a)

(9) Answer the following short questions: *Explain* your answer!

(a) Let $f(x, y, z) = y - x$. Then the line integral of $\nabla f$ around the unit circle $x^2 + y^2 = 1$ in the $xy$ plane is $\pi$, the area of the circle.

False. $\nabla f$ is a conservative field by definition, and therefore the integral over any closed curve must be 0.

(b) In the plane, if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on the disk defined by $x^2 + y^2 < 1$, then

$$\int_C \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0$$

where $C$ is the circle of radius 1/2 centered at the origin, oriented counterclockwise.

True. Denote $\frac{\partial u}{\partial y} = f$, $-\frac{\partial u}{\partial x} = g$.

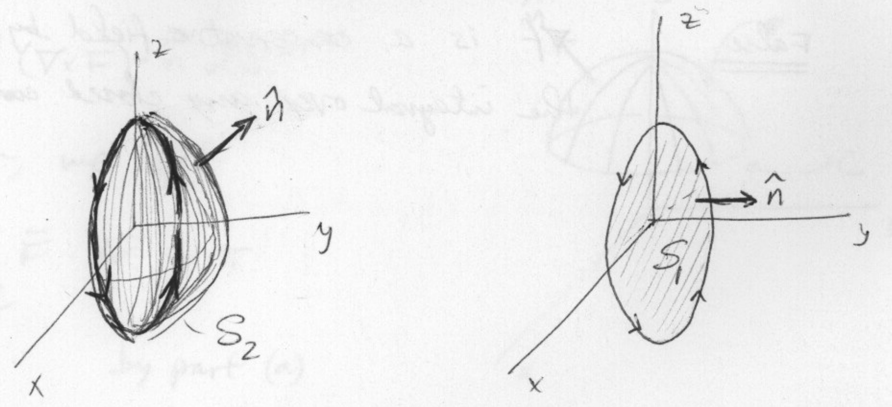
Then: $\int_C \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = \int_C f\,dx + g\,dy \overset{\text{Green}}{=} \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \circledast$

Now plug in the definitions of $g, f$, to get:

$$\circledast = \iint_D \left(-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) dA = -\iint_D \underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{0} dA = 0$$

(10) Let $C$ be the unit circle $x^2 + z^2 = 1, y = 0$, oriented counterclockwise when looking down the positive $y$ axis. Let $S_1$ be the surface $x^2 + z^2 \leq 1, y = 0$ and let $S_2$ be the surface $x^2 + y^2 + z^2 = 1, y \geq 0$.

(a) Draw a figure showing orientations for $S_1, S_2$ that are *compatible* with the orientation of $C$.




(b) (state any theorem you use, *including* its formula) For $\mathbf{F}(x,y,z) = y\mathbf{i} - z\mathbf{j} + yz^2\mathbf{k}$, show that

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Use Stokes' Theorem for both $S_1$ and $S_2$, as both of them have $C$ as their boundary:

$$\iint_{S_1} (\nabla\times\vec{F})\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \iint_{S_2} (\nabla\times\vec{F})\cdot d\vec{S}$$

(c) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Let us find $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -z & yz^2 \end{vmatrix} = (z^2+1)\,\hat{i} - \hat{k}$

Consider the surface $S_1$ ; $\hat{n} = (0,1,0)$. Therefore $(\nabla\times\vec{F})\cdot\hat{n} = 0$ on $S_1$.

Using Stokes: $\int_C \vec{F}\cdot d\vec{r} = \iint_{S_1} (\nabla\times\vec{F})\cdot\hat{n}\, dS = 0$.

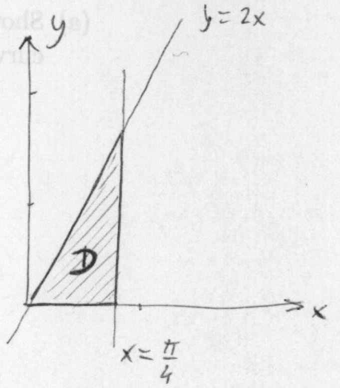
(11) Let $W$ be the region in space under the graph of

$$f(x,y) = \cos y \cdot e^{1-\cos 2x} + xy$$

over the region in the $xy$ plane bounded by the line $y = 2x$, the $x$ axis, and the line $x = \pi/4$.

(a) Find the volume of $W$.

$$\mathrm{Vol}(W) = \iint_D f(x,y)\, dA = \int_0^{\pi/4}\int_0^{2x} \left(\cos y\, e^{1-\cos 2x} + xy\right) dy\, dx$$




$$= \int_0^{\pi/4} \left( \sin y \Big|_0^{2x} e^{1-\cos 2x} + x \frac{y^2}{2}\Big|_0^{2x} \right) dx$$

$$= \int_0^{\pi/4} \left(\sin 2x\, e^{1-\cos 2x} + 2x^3\right) dx$$

$$= \underbrace{\int_0^{\pi/4} \sin 2x\, e^{1-\cos 2x}\, dx} + \frac{x^4}{2}\Big|_0^{\pi/4} = \int e^u \frac{du}{2} + \frac{\pi^4}{2\cdot 4^4}$$

$u = 1 - \cos 2x$

$du = 2\sin 2x\, dx$

$$= \frac{1}{2} e^u + \frac{\pi^4}{2\cdot 4^4} = \frac{1}{2} e^{1-\cos 2x}\Big|_0^{\pi/4} + \frac{\pi^4}{2\cdot 4^4}$$

$$= \frac{1}{2} e^{1-\cos\frac{\pi}{2}} - \frac{1}{2} e^{1-\cos 0} + \frac{\pi^4}{2\cdot 4^4}$$

$$= \frac{1}{2} e^1 - \frac{1}{2} e^0 + \frac{\pi^4}{2\cdot 4^4} = \frac{1}{2}(e-1) + \frac{\pi^4}{2\cdot 4^4}$$

(b) Let $\mathbf{F} = 5x\mathbf{i} + 5y\mathbf{j} + 5z\mathbf{k}$ be the velocity field of a fluid in space. Calculate the rate at which fluid is leaving the region $W$.

We want to use the divergence theorem: $\iiint_W \nabla \cdot \vec{F}\, dV = \iint_\sigma \vec{F} \cdot d\vec{S} = ?$

where $\sigma$ is the surface enclosing $W$.

Easier to calculate the LHS: $\nabla \cdot \vec{F} = 5 + 5 + 5 = 15$.

$$\Rightarrow \iint_\sigma \vec{F} \cdot d\vec{S} = \iiint_W 15\, dV = 15\, \mathrm{Vol}(W) = 15 \times (\quad)$$

(12) Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where

$$P(x,y,z) = \frac{-y}{x^2+y^2}, \quad Q(x,y,z) = \frac{x}{x^2+y^2}, \quad R(x,y,z) = z.$$

(a) Show that $\nabla \times \mathbf{F} = \mathbf{0}$ away from the $z$ axis.

Need to show that $\nabla \times \vec{F} = \vec{0}$ when $(x,y) \neq (0,0)$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + \Big[\underbrace{\frac{\partial}{\partial x}\Big(\frac{x}{x^2+y^2}\Big)}_{\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}} - \underbrace{\frac{\partial}{\partial y}\Big(-\frac{y}{x^2+y^2}\Big)}_{\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}}\Big]\hat{k}$$

$$\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2} = 0$$

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$ is not zero if the curve $C$ is the circle $x^2 + y^2 = 4$, $z = 0$ (choose your favorite orientation).

Parametrize: $C: \ (2\cos t, 2\sin t, 0) \quad 0 \le t \le 2\pi$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \Big(\frac{-2\sin t}{4}, \frac{2\cos t}{4}, 0\Big) \cdot (-2\sin t, 2\cos t, 0)\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi \neq 0.$$

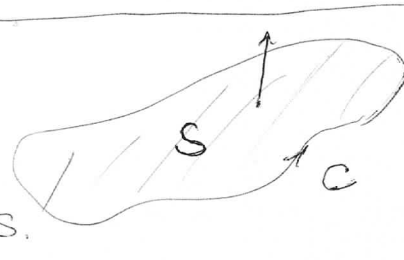
(c) Can you apply Stokes' Theorem to the line integral appearing in (b)? Explain your answer.

We cannot apply Stokes' Theorem, since it requires $\vec{F}$ to be continuous, as well as its first partial derivatives on the surface $S$. Since any surface $S$ that has $C$ as its boundary must intersect the $z$-axis, this condition doesn't hold, as $\vec{F}$ isn't defined on the $z$-axis.

Stokes' Theorem states:



$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$




But only when $\vec{F}$ is "smooth" throughout $S$.