Lectures on Harmonic Analysis and PDEs, Cardiff

Dynamics of nonlinear Schrödinger equations

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Introduction

- Consider Cauchy problem for nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^{p-1} u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{0.1}$$

$u(t,x): \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\mu = \pm 1$ corresponding to defocusing/focusing case. There are three important conserved quantities:

$$\text{Mass}: \ M(u) = \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx = M(u_0);$$

$$\text{Energy}: \ E(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx + \frac{\mu}{1+p} |u(t,x)|^{p+1} \, dx = E(u_0);$$

$$\text{Momentum}: \ P(u) = \mathrm{Im} \int_{\mathbb{R}^d} \nabla u \bar{u} \, dx = P(u_0).$$

- **Equation (0.1) admits a number of symmetries in energy space** H^1:

1. **Space-time translation invariance:** if $u(t,x)$ solves (0.1), then so does $u(t+t_0, x+x_0),\ (t_0,x_0) \in \mathbb{R}\times\mathbb{R}^d$;
2. **Phase invariance:** if $u(t,x)$ solves (0.1), then so does $e^{i\gamma}u(t,x),\ \gamma\in\mathbb{R}$;
3. **Galilean invariance:** if $u(t,x)$ solves (0.1), then for $\beta\in\mathbb{R}^d$, so does $e^{i\frac{\beta}{2}\cdot(x-\frac{\beta}{2}t)}u(t,x-\beta t)$;
4. **Scaling invariance:** if $u(t,x)$ solves (0.1), then so does $u_\lambda(t,x)$ defined by

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x), \quad \lambda > 0. \tag{0.2}$$

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- For $s_c = \frac{d}{2} - \frac{2}{p-1}$, $\left\|u_\lambda(t,\cdot)\right\|_{\dot{H}^{s_c}_x(\mathbb{R}^d)} = \|u(\lambda^2 t,\cdot)\|_{\dot{H}^{s_c}_x(\mathbb{R}^d)}$,

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- For $s_c = \frac{d}{2} - \frac{2}{p-1}$, $\left\|u_\lambda(t,\cdot)\right\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} = \|u(\lambda^2 t, \cdot)\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)}$,
  - For $s_c = 0$ we call the problem (0.1) **mass critical**($p = 1 + \frac{4}{d}$),
  - For $s_c = 1$, we call the problem (0.1) **energy critical**($p = 1 + \frac{4}{d-2}$),
  - For $s_c \in (0,1)$, **interpolate between mass and energy critical**,
  - For $s_c > 1$ we call the problem (0.1) **energy supercritical**($p > 1 + \frac{4}{d-2}$).

Definition 0.1 (Classical, Weak, strong, strong Strichartz solution)

- **Classical solution:** A function u is a *classical* solution of (0.1) on a time interval I containing 0 if $u \in C_t^1(I, C_x^2(\mathbb{R}^d))$ and solves (0.1) in the classical sense.
- **Weak solution:** A function u is a *weak* solution of (0.1) if $(\partial_t u, \nabla_x u) \in L_t^\infty(\mathbb{R}, L_x^2)$, $u \in L_t^\infty(\mathbb{R}, H^1) \cap L_t^\infty L_x^{p+1}$, and u solves (0.1) in distribution sense, namely

$$-i \iint_{\mathbb{R}\times\mathbb{R}^d} u\partial_t\varphi \, dx \, dt + \iint_{\mathbb{R}\times\mathbb{R}^d} u\Delta\varphi \, dx \, dt + \iint_{\mathbb{R}\times\mathbb{R}^d} f(u)\varphi \, dx \, dt$$
$$= 0, \quad \forall \quad \varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d), \tag{0.3}$$

and the energy inequality

$$E(u(t)) \leq E(u(0)), \quad \forall \quad t \in \mathbb{R} \tag{0.4}$$

holds. Here $f(u) = \pm|u|^{p-1}u$.

• **Strong solution:** $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a *(strong) solution* of (0.1) with data $u_0 \in H^s$, $s \in \mathbb{R}$, if u satisfies

$$u(t,\cdot) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}\Big(\mu|u|^{p-1}u\Big)(s)\,ds \tag{0.5}$$

for t in time interval I containing 0.

•**Strong Strichartz solution:** we say that u is a *strong Strichartz solution* of (0.1), if u is a strong solution and u belongs to some auxiliary spaces associated with the Strichartz estimate, such as some spatial-time space $L_t^q(I, L_x^r(\mathbb{R}^d))$.

Classical → Strong Strichartz → Strong → weak

Basic mathematical problems in Nonlinear PDEs

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Basic mathematical problems in Nonlinear PDEs

• **Wellposedness**: Existence, uniqueness, continuous dependence on the data, persistence of regularity.

• **Scattering theory**: If the solutions exist for all $t \in \mathbb{R}$, does it approach a free solution? Critical norm conjecture.

• **Blow-up dynamics**: If the solution breaks down in finite time, can we describe the mechanism by which it dose so? For example, via energy concentration at the tip of a light cone? Usually, symmetries play a crucial role. How about the qualitative description of **singularity formation** and blowup rate? Solitary wave conjecture.

• **Special solution**: If the solution does not approach a free solution, does it scatter to something else? A stationary nonzero solution, for example? Some physical equations exhibit nonlinear bound states, which represent elementary particles.
• **Stability theory**: If special solutions exist such as stationary or time-periodic solutions, are they orbitally stable? are they asymptotically stable?
• **Multi-bump solutions**: Is it possible to construct solutions which asymptotically split into moving " solitons " plus radiation? Lorentz invariant dictates the dynamics of single solitons. , **solitary resolution conjecture**
• **Resolution into Multi-bumps**: Do all solutions decompose in this fashion ? suppose solutions exist for all $t \geq 0$: Either scatter to a free wave, or the energy collects in "pockets" formed by such "soltons" ? Quantization of energy.

Gaussian kernel for heat equation

Let $e^{t\Delta}$ be the free heat operator, given by

$$[e^{t\Delta}f](x) = G_t(x) * f, \quad G_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4\pi t}}, \quad t > 0. \tag{1.1}$$

Theorem 1

Let $\phi \in L^1(\mathbb{R}^d)$, and $\int_{\mathbb{R}^d} \phi(x)\, dx = 1$, $\psi(x) = \sup_{|y|\geq|x|} |\phi(y)|$, $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$. If $\psi(x) \in L^1(\mathbb{R}^d)$, then for any $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq +\infty$, we have

$$\lim_{\varepsilon\to 0} \phi_\varepsilon * f = f(x), \quad \forall\, x \in L_f,$$

where $L_f := \{x : \lim_{r\to 0} \frac{1}{r^d} \int_{|y|\leq r} |f(x-y) - f(x)|\, dy = 0\}$, Lebesgue point. Moreover, by the fact that (Lebesgue differentiation theorem), for any $f \in L^p$, almost every point x is a Lebesgue point! Thus,

$$\lim_{\varepsilon\to 0} \phi_\varepsilon * f = f(x), \quad \textbf{\textit{a.e.}}\ x \in \mathbb{R}^d.$$

$\lim_{t\to 0+} G_t(x) = \delta(x), \qquad$ **in** $\quad \mathcal{D}'(\mathbb{R}^d)$.

Dispersive estimate

- Let $e^{it\Delta}$ be the free Schrödinger propagator, given by

$$[e^{it\Delta}f](x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y)\,dy, \quad \text{for } t \neq 0. \tag{1.2}$$

From this explicit formula we can read off the dispersive estimate

$$\|e^{it\Delta}f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|f\|_{L_x^1(\mathbb{R}^d)} \quad \text{for } t \neq 0.$$

Interpolating with $\|e^{it\Delta}f\|_{L_x^2(\mathbb{R}^d)} \equiv \|f\|_{L_x^2(\mathbb{R}^d)}$ then yields

$$\left\|e^{it\Delta}f\right\|_{L_x^r(\mathbb{R}^d)} \leq C|t|^{-d(\frac{1}{2}-\frac{1}{r})}\|f\|_{L_x^{r'}(\mathbb{R}^d)}, \quad \text{for } t \neq 0. \tag{1.3}$$

and $2 \leq r \leq \infty$, where $\frac{1}{r} + \frac{1}{r'} = 1$.

Fraunhofer formula:

$$\lim_{t\to\pm\infty} \left\| e^{it\Delta}\phi - e^{\mp id\pi/4} \frac{e^{\pm i\frac{|x|^2}{4t}}}{(4\pi t)^{d/2}} \hat{\phi}\Big(\pm \frac{x}{4\pi|t|}\Big) \right\|_{L_x^2(\mathbb{R}^d)} = 0. \tag{1.4}$$

- The **pointwise convergence** problem is to ask what the minimal s is to ensure

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x), \quad \text{a.e. } x, \quad \forall f \in H^s(\mathbb{R}^d). \tag{1.5}$$

Equivalently,

$$\Big\| \sup_{|t|<1} |e^{it\Delta} f| \Big\|_{L^2(B(0,1))} \le C_{s,d} \|f\|_{H^s(\mathbb{R}^d)}. \tag{1.6}$$

Pointwise convergence

$d=1$	$s \ge \frac{1}{4}$	Carleson [23]	Dahlberg-Kenig[46]$s < \frac{1}{4}$ fails
$d=2$	$s = \frac{3}{8}+$	S.Lee[126], S.Shao[214]	
	$s > \frac{1}{3}$	X.Du, L. Guth, X. Li[60]	Bourgain [19]$s < \frac{1}{3}$ fails
$d \ge 3$	$s > \frac{1}{2}$	Sjölin[216], Vega[234]	
	$s > \frac{1}{2} - \frac{1}{4d}$	Bourgain[18]	$s < \frac{1}{2} - \frac{1}{d}$ fails in $d \ge 5$
		Luca-Rogers[138],DG[47]	$s < \frac{1}{2} - \frac{1}{d+2}$ fails in $d \ge 5$
		Bourgain [19]	$s < \frac{1}{2} - \frac{1}{2(d+1)}$
$d \ge 1$	$s \ge \frac{1}{4}$	Gigante-Soria[80]	radial initial data

Strichartz estimate

$$\|e^{it\Delta}f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}.$$

Proposition 1.1 (Strichartz estimate, Ginibre-Velo[82], Strichartz[220], Keel-Tao[96])

Suppose $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a solution to $(i\partial_t + \Delta)u = h$. Then, for $2 \leq q, r \leq \infty$ and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and $(q, r, d) \neq (2, \infty, 2)$ (Denote by $(q, r) \in \Lambda$ for simplicity),

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|u(t_0)\|_{L_x^2(\mathbb{R}^d)} + \|h\|_{L_t^{q_1'} L_x^{r_1'}(I)}$$

for every $t_0 \in I$.

1. The key ingredient for the proof of Proposition 1.1 is the estimate (1.3).
2. For the non-endpoint case, Proposition 1.1 is a direct result of (1.3) and the standard TT^* argument.
3. For the endpoint $(2, \frac{2d}{d-2})$, see Keel-Tao[96].

Application of Strichartz estimate

Theorem 1.2

Let u be the solution to $i\partial_t u + \Delta u = |u|^2 u$ with $u(0,x) = u_0(x) \in H^1(\mathbb{R}^3)$, then, the solution is global and

$$u \in C(\mathbb{R},\ H^1(\mathbb{R}^3)) \bigcap L^4_{\mathrm{loc}}(\mathbb{R}, W^{1,3}(\mathbb{R}^3)).$$

Banach fixed point argument and Strichartz estimate.

Scattering is equivalent to scattering size to be finite.

Morawetz multiplier method

Let the multiplier

$$Au = \frac{x}{|x|} \cdot \nabla u + \frac{d-1}{2|x|} u,$$

and u solves $i\partial_t u + \Delta u = 0$. Then

$$\frac{d}{dt} \langle Au, u \rangle_{L_x^2} = i \Big\langle [A, \Delta] u, u \Big\rangle, \ \Big| \langle Au, u \rangle_{L_x^2} \Big| \leq C \| u(t, \cdot) \|^2_{\dot{H}^{\frac{1}{2}}}.$$

A simple computation shows the commutator

$$[A, \Delta] = \begin{cases} -2\Delta_{\mathbb{S}^2} |x|^{-3} + c\delta(x) & \text{if } d = 3, \\ -2\Delta_{\mathbb{S}^{d-1}} |x|^{-3} + \frac{1}{2}(d-1)(d-3)|x|^{-3} & \text{if } d \geq 4. \end{cases} \tag{1.7}$$

$$-i \langle Au, u \rangle_{L_x^2} \Big|_{t_1}^{t_2} = 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|\nabla_\theta u|^2}{|x|^3} \, dx \, dt + \begin{cases} 2 \int_{t_1}^{t_2} |u(t,0)|^2 \, dt & \text{if } \quad d = 3, \\ \frac{1}{2}(d-1)(d-3) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^3} \, dx \, dt & \text{if } \quad d \geq 4. \end{cases}$$

For $i\partial_t u + \Delta u = \mu |u|^{p-1} u$, additional term $\frac{p-1}{p+1} \mu \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|u(t,x)|^{p+1}}{|x|} \, dx \, dt$.

Morawetz estimate

Let u solve

$$iu_t + \Delta u = \mathcal{N}, \tag{1.8}$$

Define Morawetz action at 0 as following

$$M_a^0(t) = 2\int_{\mathbb{R}^d} \partial_j a(x) Im(\bar{u}\partial_j u)dx. \tag{1.9}$$

A simple computation shows

Lemma 1.3 (Morawetz's identity)

$$\frac{d}{dt}M_a^0(t) = \int_{\mathbb{R}^d}(-\Delta\Delta a)|u|^2dx + 4\int_{\mathbb{R}^d} a_{jk}\Re(\partial_j\bar{u}\partial_k u)dx + 2\int_{\mathbb{R}^d} a_j(x)\{\mathcal{N}, u\}_p^j dx, \tag{1.10}$$

where $\{f, g\} = \Re(f\nabla\bar{g} - g\nabla\bar{f})$.

Remark 1.1 (Morawetz's identity-I)

If $a(x) = |x|$, and $\mathcal{N} = |u|^{p-1}u$, then

$$\begin{cases} a_j(x) = \frac{x_j}{|x|},\ a_{jk}(x) = \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3},\quad \Delta a = \frac{d-1}{|x|},;\ \{\mathcal{N}, u\}_p^j = -\frac{p-1}{p+1}\partial_j(|u|^{p+1}) \\ -\Delta\Delta a = 4\pi\delta(x), \quad d = 3, \\ -\Delta\Delta a = \frac{(d-1)(d-3)}{|x|^3},\ d \geq 4. \end{cases}$$

Hence,

$$\frac{d}{dt}M_a^0(t) = \int_{\mathbb{R}^d}(-\Delta\Delta a)|u(x)|^2 dx + 4\int_{\mathbb{R}^d}\frac{|\not\nabla_0 u|^2}{|x|}dx + \frac{2(p-1)(d-1)}{p+1}\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{|x|}dx,$$

where $\not\nabla_0$ denotes the complement of the radial portion of the gradient.

$$\not\nabla_0 u = \nabla u - \frac{x}{|x|}\Big(\frac{x}{|x|}\cdot\nabla u\Big),\ |\not\nabla_0 u|^2 = |\nabla u|^2 - \Big|\frac{x}{|x|}\cdot\nabla u\Big|^2 \triangleq |\nabla u|^2 - |\nabla_{r,0} u|^2.$$

By duality argument, and Hardy inequality, we obtain

$$|M_a^0(t)| \leq \Big|\int \frac{x}{|x|}\cdot u\nabla u dx\Big| \leq C\|u\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Theorem 1.4 (The classical Morawetz estimate, Lin-Strauss)

If $d \geq 3$, $u: I \times \mathbb{R}^d \to \mathbb{C}$ *solves* $i\partial_t u + \Delta u = |u|^{p-1}u$, *then, there holds*

$$\int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^{p+1}}{|x|} \, dx \, dt \lesssim \|u\|^2_{L_t^\infty \dot{H}^{\frac{1}{2}}} \tag{1.11}$$

Theorem 1.4 (The classical Morawetz estimate, Lin-Strauss)

If $d \geq 3$, $u : I \times \mathbb{R}^d \to \mathbb{C}$ solves $i\partial_t u + \Delta u = |u|^{p-1}u$, then, there holds

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Proposition 1.5 (Space-localized Morawetz estimate)

Let $p > 3$ and $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be a solution to $i\partial_t u + \Delta u = |u|^{p-1}u$. Then for $C \geq 1$, we have

$$\int_I \int_{|x| \leq C|I|^{1/2}} \frac{|u(t,x)|^{p+1}}{|x|}\, dx\, dt \lesssim (C|I|^{1/2})^{2s_c - 1} \left\{ \|u\|^2_{L_t^\infty \dot{H}_x^{s_c}(I\times\mathbb{R}^3)} + \|u\|^{p+1}_{L_t^\infty \dot{H}_x^{s_c}(I\times\mathbb{R}^3)} \right\}, \tag{1.12}$$

with $s_c = \frac{3}{2} - \frac{2}{p-1}$.

Interaction Morawetz estimate

- Interaction Morawetz identity [40]. For a fixed weight $a : \mathbb{R}^d \to \mathbb{R}$ and a function φ solving

$$(i\partial_t + \Delta)\varphi = \mathcal{N},$$

Morawetz action at y

$$M_a^y(t) = 2\int_{\mathbb{R}^d} \partial_j a(x-y) Im(\bar{\varphi}\partial_j\varphi)dx.$$

Interaction Morawetz potential

$$M(t) = 2\Im\int_{\mathbb{R}^d} M_a^y(t)|\varphi(y)|^2 dy = 2\Im\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi(y)|^2 a_k(x-y)(\varphi_k\bar{\varphi})(x)\,dx\,dy, \tag{1.13}$$

Mass bracket and momentum bracket

$$\begin{cases} \{f,g\}_m := \Im(f\bar{g}), & \text{Mass bracket} \\ \{f,g\}_P := \Re(f\nabla\bar{g} - g\nabla\bar{f}), & \text{Mmomentum bracket.} \end{cases}$$

Proposition 1.6 (Morawetz identity)

$$\partial_t M(t) = -\iint |\varphi(y)|^2 a_{jjkk}(x-y)|\varphi(x)|^2\, dx\, dy \tag{1.14}$$

$$+ \iint 4a_{jk}(x-y)\Big[|\varphi(y)|^2 \Re(\bar{\varphi}_k \varphi_j)(x) - \Im(\bar{\varphi}\varphi_j)(y)\Im(\bar{\varphi}\varphi_k)(x)\Big] dx\, dy \tag{1.15}$$

$$+ \iint \{\mathcal{N}, \varphi\}_m(y) 4a_k(x-y)\Im\{\bar{\varphi}\varphi_k)(x)\, dx\, dy \tag{1.16}$$

$$+ \iint |\varphi(y)|^2 2\nabla a(x-y)\cdot \{\mathcal{N}, \varphi\}_P(x)\, dx\, dy. \tag{1.17}$$

Interaction Morawetz estimate ($d \geq 3$) Choosing weight $a(x) = |x|$, we have

$$\int_I \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\varphi(t,y)|^2 |\varphi(t,x)|^2}{|x-y|^3} dx\, dy\, dt \lesssim \|\varphi\|^3_{L^\infty_t L^2_x(I\times\mathbb{R}^d)} \|\nabla\varphi\|_{L^\infty_t L^2_x(I\times\mathbb{R}^d)}.$$

Interaction Morawetz estimate ($d = 1, 2$). [40, 196, 32, 33]

$$\left\||\nabla|^{\frac{3-d}{2}}(|u|^2)\right\|^2_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|^2_{L^2_x} \|u\|^2_{L^\infty(\mathbb{R};\dot{H}^{\frac{1}{2}})}, \quad d \geq 1. \tag{1.18}$$

A simple example of interaction Morawetz estimate

One can use this to give a simple proof of Ginibre-Velo about the defocusing energy-subcritical NLS. In fact, we take 3d cubic NLS $i\partial_u + \Delta u = |u|^2 u$ for example. By interaction Morawetz estimate,

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \le C(M,E).$$

Interpolating this with $\|u\|_{L^\infty_t L^6_x} \le C\|u\|_{L^\infty_t \dot{H}^1} \le CE(u_0)$ implies

$$\|u\|_{L^6_t L^{\frac{9}{2}}_x(\mathbb{R}\times\mathbb{R}^3)} \le C(E,M).$$

This implies scattering. Indeed, letting the aympotic state

$$u_+(x) = u_0(x) + i\int_0^{+\infty} e^{is\Delta}(|u|^2u)(s)\,ds,$$

$$\|u - e^{it\Delta}u_+\|_{H^1} \lesssim \left\| \int_t^{+\infty} e^{i(t-s)\Delta}(|u|^2u)(s)\,ds \right\|_{H^1} \lesssim \left\| \langle\nabla\rangle(|u|^2u)\right\|_{L^2_t L^{\frac{6}{5}}_x([t,+\infty)\times\mathbb{R}^3)}$$
$$\lesssim \left\|\langle\nabla\rangle u\right\|_{L^6_t L^{\frac{18}{7}}_x} \|u\|_{L^6_t L^{\frac{9}{2}}_x([t,+\infty]\times\mathbb{R}^3)} \to 0, \quad t\to+\infty.$$

Local smoothing estimate

- Let $u(t,x) = e^{it\Delta}f$. Then

$$\sup_{x_0\in\mathbb{R}^d, R>0} \frac{1}{R}\int_{\mathbb{R}}\int_{B(x_0,R)} |\nabla u(t,x)|^2\, dx\, dt \simeq C\|f\|^2_{\dot H^{\frac12}}, \quad \forall f\in \dot H^{\frac12}(\mathbb{R}^d,$$

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}} \frac{|\nabla u(t,x)|^2}{|x|^a}\, dt\, dx \le C\|f\|^2_{\dot H^{\frac a2}(\mathbb{R}^d)}, & a>1; \\ \displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{|\nabla u(t,x)|^2}{\langle x\rangle^{1+\delta}}\, dx\, dt \le C\|f\|_{\dot H^{\frac12}(\mathbb{R}^d)}, & \forall\ \delta>0. \end{cases} \tag{1.19}$$

Lemma 1.7 (B.Simon-Local smoothing estimate[215])

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} \frac{\left||\nabla|^{\frac12} e^{it\Delta}f(x)\right|^2}{\langle x\rangle}\, dt\, dx \le \frac{\pi}{2}\|f\|^2_{L^2(\mathbb{R}^d)}, \quad d\ge 3 \tag{1.20}$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} \frac{|e^{it\Delta}f(x)|^2}{|x|^2}\, dt\, dx \le \frac{\pi}{d-2}\|f\|^2_{L^2(\mathbb{R}^d)}, \quad d\ge 3. \tag{1.21}$$

Lemma 1.8 (Ben-Artzi and Klainerman-local smoothing estimate [6])

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} \frac{\left|(1-\Delta)^{\frac{1}{4}} e^{it\Delta} f(x)\right|^2}{\langle x\rangle}\, dt\, dx \le C\|f\|^2_{L^2(\mathbb{R}^d)}, \quad d \ge 3. \tag{1.22}$$

The above Kato smoothing estimate is sharp. In fact, for every $\delta > 0$,

$$\left\|P_{\le 1}(-\Delta)^{\frac{1}{4}} e^{it\Delta}(e^{-|x|^2} e^{ix\cdot v})\right\|_{L^2_{t,x}} \sim 1,$$

while

$$\sup_{v\in\mathbb{R}^d}\left\|P_{\le 1}(-\Delta)^{\frac{1}{4}+\delta} e^{it\Delta}(e^{-|x|^2} e^{ix\cdot v})\right\|_{L^2_{t,x}} = +\infty. \tag{1.23}$$

Proposition 1.9 (Planchon-Vega[196])

Let u be a solution to the linear Schrödinger equation on $\mathbb{R}$:

$$\sup_x \int_{\mathbb{R}} |\partial_x u|^2(x,t)\, dt = 4\pi\|u_0\|^2_{\dot H^{\frac{1}{2}}}. \tag{1.24}$$

Theorem 2 (Kato smoothing for radial functions, Li-Zhang[132])

Let the dimension $d \geq 2$. Then for any radial function f on $\mathbb{R}^d$, we have

$$\left\| |x|^{\frac{d-1}{2}} |\nabla|^{\frac{1}{2}} e^{it\Delta} f \right\|_{L_x^\infty L_t^2(\mathbb{R}^d \times \mathbb{R})} \lesssim \|f\|_{L_x^2}, \tag{1.25}$$

and

$$\left\| |x|^{\frac{d-1}{2}} |\nabla| \int_{\mathbb{R}} e^{i(t-s)\Delta} |y|^{\frac{d-1}{2}} f(s,y)\, ds \right\|_{L_x^\infty L_t^2(\mathbb{R}^d \times \mathbb{R})} \lesssim \|f\|_{L_x^1 L_t^2(\mathbb{R}^d \times \mathbb{R})}. \tag{1.26}$$

While, the following estimate

$$\left\| |x|^{\frac{d-1}{2}} |\nabla| \int_0^t e^{i(t-s)\Delta} |y|^{\frac{d-1}{2}} f(s,y)\, ds \right\|_{L_x^\infty L_t^2(\mathbb{R}^d \times \mathbb{R})} \lesssim \|f\|_{L_x^1 L_t^2(\mathbb{R}^d \times \mathbb{R})} \tag{1.27}$$

fails in general. **An ϵ-version of (1.27) holds, namely**

$$\left\| |x|^{\frac{d-1-\epsilon}{2}} |\nabla|^{1-\epsilon} \int_0^t e^{i(t-s)\Delta} |y|^{\frac{d-1-\epsilon}{2}} f(s,y)\, ds \right\|_{L_x^\infty L_t^2(\mathbb{R}^d \times \mathbb{R})} \lesssim \|f\|_{L_x^1 L_t^2(\mathbb{R}^d \times \mathbb{R})}.$$

Blow up for focusing NLS

Symmetries not in energy space H^1:

• Pseudo-conformal transformation: if $u(t,x)$ solves (0.1), then

$$v(t,x) = \frac{1}{|t|^{d/2}} \bar{u}\Big(\frac{1}{t}, \frac{x}{t}\Big) e^{i\frac{|x|^2}{4t}}.$$

This additional symmetry yields the conservation of the pseudo-conformal energy for initial data $u_0 \in \Sigma := H^1 \cap \{xu_0 \in L^2\}$, which is most frequently expressed as

$$\begin{aligned}\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 =& 4\frac{d}{dt}\mathrm{Im}\int x\cdot\nabla u\bar{u}(t,x)\\ =& 16E(u_0) + 4\mu\Big(d - \frac{2(d+2)}{p+1}\Big)\int_{\mathbb{R}^d}|u(t,x)|^{p+1}\,dx.\end{aligned}$$

Review the argument of Glassey

We review the classic argument of Glassey for blow-up in finite time. Thus, assume $u_0 \in H^1,\ xu_0 \in L^2$. Let I be the maximal interval of existence. For $t \in I$,

$$y(t) := \int |x|^2 |u(t)|^2 dx,\ y'(t) = 4\mathrm{Im} \int \bar{u} \nabla u \cdot x,$$

$$y''(t) = 16E(u_0) + 4\mu\Big(d - \frac{2(d+2)}{p+1}\Big) \int_{\mathbb{R}^d} |u(t,x)|^{p+1}\, dx,$$

Thus,

$$y(t) \leq 8E(u_0)t^2 + y'(0)t + y(0),$$

which implies I is finite if

$$(i) E(u_0) < 0; (ii) E(u_0) = 0, y'(0) < 0; (iii) E(u_0) > 0, y'(0) \leq -4\sqrt{2} E(u_0) y(0).$$

$$\|u_0\|^2_{L^2_x} = \|u(t)\|^2_{L^2} \leq \big\| |x| u(t) \big\|_{L^2} \|u(t)\|_{\dot{H}^1}.$$

Mass critical NLS

Mass critical NLS

- Consider

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^{\frac{4}{d}} u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.28}$$

$u(t,x): \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\mu = \pm 1$ corresponding to defocusing/focusing. For notational purpose, we shall introduce the following invariant:

$$E^G(u) = E(u) - \frac{1}{2}\Big(\frac{\mathrm{Im}(\int \partial_x u \bar{u})}{\|u\|_{L_x^2}} \Big)^2. \tag{1.29}$$

Symmetries in energy space H^1:

- Translation invariance: if $u(t,x)$ solves (1.28), then so does $u(t+t_0, x+x_0)$;
- Phase invariance: if $u(t,x)$ solves (1.28), then so does $e^{i\gamma} u(t,x)$, $\gamma \in \mathbb{R}$;
- Galilean invariance: if $u(t,x)$ solves (1.28), then for $\beta \in \mathbb{R}^d$, so does $e^{i\frac{\beta}{2}\cdot(x - \frac{\beta}{2}t)} u(t,x)$;
- Scaling invariance: if $u(t,x)$ solves (1.28), then so does $u_\lambda(t,x) = \lambda^{d/2} u(\lambda x, \lambda^2 t)$,

$\lambda > 0$, and by direct computation $\|u_\lambda(0,x)\|_{L_x^2} = \|u_0\|_{L_x^2}$.

Defocusing case: $\mu = 1$

- For any $u_0 \in L^2(\mathbb{R}^d)$, the solution to (1.28) is global and scatters in the sense that there exists $u_\pm \in L^2(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty} \|u(t,x) - e^{it\Delta}u_\pm\|_{L^2(\mathbb{R}^d)} = 0. \tag{1.30}$$

Defocusing mass-critical Schrödinger equation

	$d = 1$	$d = 2$	$d \geq 3$
radial		Killip-Tao-Visan [106]	TVZ [229, 230]
nonradial	Dodson[53]	Dodson[52]	Dodson[51]

Difficult: The lifespan-time depends not only the norm of $\|u_0\|_{L^2}$ but also the profile of the initial data u_0.

Outline of proof: The proof follows the concentration-compactness approach [8, 98], see also [107, 200, 194]. We argue by contradiction. The failure of scattering result would imply the existence of very special class of solutions. But these critical elements have so many good properties that they do not exist. Thus we get a contradiction.

We will make some further reductions, the main property of the critical elements (special counterexamples) is almost periodicity modulo symmetries:

Definition 1.10 (Almost periodic solutions)

A solution u to (1.32) with lifespan I is called *almost periodic* (*modulo symmetries*) if there exist (possibly discontinuous) functions $N: \ I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^d$, $\xi : \ I \to \mathbb{R}^d$ and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$\int_{|x-x(t)|\geq \frac{C(\eta)}{N(t)}} \left|u(t,x)\right|^2 dx + \int_{|\xi-\xi(t)|\geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta. \tag{1.31}$$

Where we call

1. $N(t)$ the *frequency scale function*;
2. $\xi(t)$ is the *frequency center function*, and $x(t)$ the *spatial center function*;
3. $C(\eta)$ the *compactness modulus function*.

Remark 1.2

• The Arzelà-Ascoli theorem tells us that a family of functions $\mathcal{F}$ is precompact in $L^2(\mathbb{R}^d)$ if and only if it is norm-bounded and there exists a compactness modulus function $C(\eta)$ such that

$$\int_{|x|\geq C(\eta)} |f(x)|^2\, dx + \int_{|\xi|\geq C(\eta)} |\hat{f}(\xi)|^2\, d\xi \leq \eta$$

uniformly for $f \in \mathcal{F}$. Thus we see that a solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ is almost periodic if and only if

$$\Big\{u(t) : t \in I\Big\} \subset \Big\{\lambda^{d/2} f(\lambda(x + x_0)) : \lambda \in (0, \infty),\ x_0 \in \mathbb{R}^d, \text{ and } f \in K\Big\}$$

for some compact $K \subset L^2(\mathbb{R}^d)$.

• This perspective also clarifies why we use the term "almost periodic". In the radial case, $x(t) = \xi(t) \equiv 0$.

Theorem 1.11 (Three enemies, Killip,Tao,Visan,Zhang[106, 229, 230])

Suppose that the scattering result fails. Then there exists a maximal-lifespan solution $u: I \times \mathbb{R}^d \to \mathbb{C}$, which is almost periodic modulo symmetries, blows up both forward and backward in time. Moreover, we can also ensure that the lifespan I and the frequency scale function $N(t): I \to \mathbb{R}^+$ match one of the following three scenarios:

1. (**Soliton-like solution**) We have $I = \mathbb{R}$ and $N(t) = 1$ for all $t \in \mathbb{R}$.
2. (**Double high-to-low frequency cascade**) We have $I = \mathbb{R}$,
$$\liminf_{t \to -\infty} N(t) = \liminf_{t \to +\infty} N(t) = 0, \text{ and } \sup_{t \to \mathbb{R}} N(t) < +\infty.$$
3. (**Self-similar solution**) We have $I = (0, +\infty)$ and
$$N(t) = t^{-\frac{1}{2}} \text{ for all } t \in I.$$

Focusing mass critical NLS

Focusing case

- We will focus on the focusing mass critical NLS

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^{\frac{4}{d}} u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.32}$$

In this case, special solutions play an important role. They are the so-called solitary waves such that $u(t,x) = e^{i\omega t} W_\omega(x),\ \omega > 0$, where W_ω solves

$$\Delta W_\omega + W_\omega |W_\omega|^{\frac{4}{d}} = \omega W_\omega. \tag{1.33}$$

1. For $d = 1$, there exists a unique solution in H^1 up to translation to (1.33);
2. For $d \geq 2$, there are infinitely many solutions with growing L^2-norm for $d \geq 2$, but there is a unique positive solution Q_ω to (1.33) up to scaling translation, Berestycki-Lions[8], Gidas-Ni- Nirenberg[76], Kwong[123];
3. Q_ω is in addition radially symmetric, letting $Q_\omega(x) = \omega^{d/4} Q(\omega^{1/2} x)$, from scaling property, we know that $Q(x)$ is a unique positive solution to (1.33) with $\omega = 1$. Therefore,$\|Q_\omega\|_{L^2} = \|Q\|_{L^2}$. Moreover, multiplying (1.33) by $\frac{d}{2} Q_\omega + x \cdot \nabla Q_\omega$ and integrating by parts yields the so-called Pohozaev identity $E(Q_\omega) = \omega E(Q) = 0$.

In particular, none of the three conservation laws (mass, energy, momentum of (1.32)) in H^1 sees the variation of size of the stationary solutions Q_ω. These two facts are deeply related to the criticality of the problem, that is the value $p = 1 + \frac{4}{d}$. Note that in dimension $d = 1$, Q can be written explicitly

$$Q(x) = \left(\frac{3}{\cosh^2(2x)} \right)^{1/4}.$$

Proposition 1.12 (Variational characterization of the ground state)

Let $v \in H^1$ such that

$$\int_{\mathbb{R}^d} |v|^2 dx = \int_{\mathbb{R}^d} Q^2 dx, \quad \text{and} \quad E(v) = 0,$$

then

$$v(x) = \lambda_0^{\frac{d}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0},$$

for some parameters $\lambda_0 \in \mathbb{R}_+^*$, $x_0 \in \mathbb{R}^d$, $\gamma_0 \in \mathbb{R}$.

Below threshold of ground state: $M(u_0) < M(Q)$

Scattering in $L_x^2(\mathbb{R}^d)$

If $u_0 \in L^2(\mathbb{R}^d)$, we refer to Dodson [54] and references therein by using compactness concentration, Long-time Strichartz estimate and frequency-localized interaction Morawetz estimate.

Focusing mass-critical Schrödinger equation

	$d = 1$	$d = 2$	$d \geq 3$
radial		Killip-Tao-Visan [106]	Killip-Visan-Zhang[111]
nonradial	Dodson[54]	Dodson[54]	Dodson[54]

We remark that the above condition is sharp: for $\|u_0\|_{L_x^2} \geq \|Q\|_{L_x^2}$, blow-up may occur. Indeed, since $E(Q) = 0$ and $\nabla E(Q) = -Q$, there exists $u_{0\varepsilon} \in \Sigma$ with $\|u_{0\varepsilon}\|_{L^2} = \|Q\|_{L^2} + \varepsilon$ and $E(u_{0\varepsilon}) < 0$, and the corresponding solution must blow up from virial identity.

Threshold solutions of ground state: $M(u_0) = M(Q)$

Solitary wave conjecture

Conjecture 1 (Solitary wave conjecture)

Let $d \geq 1$. For general initial data $u_0 \in L^2_x(\mathbb{R}^d)$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Then, either the corresponding solution scatters, or the non-scattering solution must be the solitary wave $e^{it}Q$ up to symmetries of the equation (1.32) (Space-time translation, phase, Galilean, scaling, conformal transformation).

	H^1	H^s	L^2
radial	KLVZ[103, 133] $d \geq 2$	LZ[129]	LZ[131] $d \geq 4$
nonradial	Merle[157] $T < +\infty$		Open

Finite time blowup

The pseudo-conformal transformation applied to the stationary solution $e^{it}Q$ yields an explicit solution

$$S(t,x) = \frac{1}{|t|^{d/2}} Q\Big(\frac{x}{t}\Big) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}, \; \|S(t)\|_{L^2} = \|Q\|_{L^2} \tag{1.34}$$

which scatters as $t \to -\infty$, and blows up at $T = 0$ at the speed

$$\|\nabla S(t)\|_{L^2} \sim \frac{1}{|t|}.$$

An essential feature of (1.34) is compact up to the symmetries of the flow, meaning that all the mass goes into the singularity formation

$$|S(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0} \text{ as } t \to 0. \tag{1.35}$$

Theorem 1.13 (Determination of minimal blow-up solutions, Merle [157])

Let $u_0 \in H^1(\mathbb{R}^d)$ and

$$\|u_0\|_{L_x^2} = \|Q\|_{L_x^2},$$

and assume the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then, there exists $\theta \in \mathbb{R}$, $\omega > 0, x_0 \in \mathbb{R}^d, x_1 \in \mathbb{R}^d$ such that for $t < T$

$$u(t,x) = \left(\frac{\omega}{T-t}\right)^{\frac{d}{2}} e^{i\theta + i|x-x_1|^2/4(t-T) - i\omega^2/(t-T)} Q\Big(\frac{\omega}{T-t}\big((x-x_1)-(T-t)x_0\big)\Big).$$

The existence of minimal elements in various settings has been a long standing open problem, mostly due to the fact that the existence of the minimal element for NLS relies entirely on the exceptional pseudo conformal symmetry.

Beyond threshold solutions of ground state: $M(u_0) > M(Q)$

■ Assume the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. By direct scaling argument, a known lower bound on the blowup rate is

$$\|\nabla u(t)\|_{L^2_x(\mathbb{R}^d)} \geq C(u_0)/\sqrt{T-t}. \tag{1.36}$$

Indeed, fixed $t \in [0, T)$, then

$$v^t(\tau, z) := \|\nabla u(t)\|_{L^2}^{-\frac{d}{2}} u\Big(t + \|\nabla u(t)\|_{L^2}^{-2}\tau, \|\nabla u(t)\|_{L^2}^{-1} z\Big)$$

also solves (1.32). It is easy to see that

$$\|\nabla v^t(0)\|_{L^2} = 1, \quad \|v^t(0)\|_{L^2} = \|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$

By the local well-posedness, we derive that there exists $\tau_0 > 0$ independent of t such that v^t is defined in $[0, \tau_0]$. Hence

$$t + \|\nabla u(t)\|_{L^2}^{-2}\tau_0 \leq T \Longrightarrow \|\nabla u(t)\|_{L^2} \geq \frac{\tau_0}{\sqrt{T-t}}.$$

Two type solutions

■ **The sharp upper and lower bound on the blow-up rate and universality of blow-up profile**. The question of the description of stable blow up bubbles has attracted a considerable attention which started in the 80's with the development of sharp numerical methods. Most results on blow-up dynamics for (1.32) deal with the perturbative situation where

$$u_0 \in \mathcal{B}_{\alpha^*} := \left\{ u_0 \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} Q^2 dx \le \int_{\mathbb{R}^d} |u_0|^2 dx < \int_{\mathbb{R}^d} Q^2 dx + \alpha^* \right\},$$

for some small (explicit) constant $\alpha^* > 0$. At least two different blow-up mechanisms are known to occur:

• In dimension $d \in \{1, 2\}$, Bourgain and Wang[20] proved that there exists a family of solutions of type $S(t)$ with blowup speed

$$\|\nabla u(t)\|_{L^2} \sim \frac{1}{T - t}.$$

• By making use of numerical simulations, Landman, Papanicolaou, Sulem and Sulem [124], and heuristic (formal) arguments in Sulem-Sulem [222], suggest existence of solutions blowing up like

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^d)} \sim \left(\frac{\ln|\ln(T-t)|}{T-t}\right)^{\frac{1}{2}}, \quad d=2.$$

In dimension $d=1$, Perelman[193] proves the existence of an even solutions of this type and its stability in some space $E \subset H^1$.

Theorem 1.14 (Bourgain-Wang solutions, [20])

Let $d=1,2$. Let u^* be such that

$$u^* \in X_A = \{f \in H^A \text{ with } (1+|x|^A)f \in L^2\} \tag{1.37}$$

$$D^\alpha u^*(0) = 0, \text{ for } 1 \leq |\alpha| \leq A, \tag{1.38}$$

for some A large enough. Then, there exists a solution $u_{\mathrm{BW}} \in C((-\infty,0),H^1)$ to (1.32) which blows up at $t=0,\ x=0$ and satisfies:

$$u_{\mathrm{BW}}(t) - S(t) \to u^* \text{ in } H^1 \text{ as } t \to 0. \tag{1.39}$$

Instability of Bourgain-Wang solutions

Theorem 1.15 (Instability of Bourgain-Wang solutions, [169])

Let $d = 2$. Let u^* satisfy (1.37) and (1.38) and let $u_{\mathrm{BW}} \in C((-\infty, 0), H^1)$ be the corresponding Bourgain-Wang solution. Then there exists a continuous map

$$\eta \in [-1, 1] \to u^\eta(-1) \in \Sigma$$

such that $u^\eta(t)$ being the solution of (1.32) with initial data $u^\eta(-1)$ at $t = -1$,

- $u^{\eta=0}(t) \equiv u_{\mathrm{BW}}(t)$;
- $\forall\ \eta \in [0, 1]$, $u^\eta \in C(\mathbb{R}, \Sigma)$ is global in time and scatters;
- $\forall\ \eta \in [-1, 0)$, $u^\eta \in C((-\infty, T^\eta), \Sigma)$ blows up in the log-log regime at $-1 < T^\eta < 0$.

The situation has been clarified by Merle and Raphaël in the series of papers [159, 160, 147, 162, 197, 163]. Let us define the differential operator

$$\Lambda = \frac{d}{2} + y \cdot \nabla,$$

which will be of constant use. Then we introduce the following property:

Spectral property. Let $d \geq 1$. Consider the two real Schrödinger operators

$$\mathcal{L}_1 = -\Delta + \frac{2}{d}\Big(\frac{4}{d} + 1\Big) Q^{\frac{4}{d}-1} y \cdot \nabla Q, \quad \mathcal{L}_2 = -\Delta + \frac{2}{d} Q^{\frac{4}{d}-1} y \cdot \nabla Q, \tag{1.40}$$

and the real quadratic form

$$H(\varepsilon, \varepsilon) = (\mathcal{L}_1 \varepsilon_1, \varepsilon_1) + (\varepsilon_2 \varepsilon_2, \varepsilon_2), \quad \text{for } \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1.$$

Then there exists a universal constant $\tilde{\delta}_1 > 0$ such that for all $\varepsilon \in H^1$, if

$$(\varepsilon_1, Q) = (\varepsilon_1, Q_1) = (\varepsilon_1, yQ) = (\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = (\varepsilon_2, \nabla Q) = 0,$$

then

$$H(\varepsilon, \varepsilon) \geq \tilde{\delta}_1 \int_{\mathbb{R}^d} \Big(|\nabla \varepsilon|^2 + |\varepsilon|^2 e^{-2^{-}|y|} \Big) dx,$$

where $Q_1 = \frac{d}{2} Q + y \cdot \nabla Q$ and $Q_2 = \frac{d}{2} Q_1 + y \cdot \nabla Q_1$.

Dynamics of mass-critical NLS

Theorem 1.16 (Dynamics of NLS, Merle-Raphael[159, 160, 147, 162, 197, 163])

Let $d \in \{1,2,3,4\}$. Then there exist $\alpha^* > 0$ and a universal constant $C^* > 0$ such that the following is true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and $u(t) \in C([0,T), H^1(\mathbb{R}^d))$ be the corresponding maximal life-span solution on right to (1.32).

(i) **Estimates on the blow-up speed:** assume $u(t)$ blows up in finite time i.e., $0 < T < +\infty$, for t close enough to T , we have either

$$\lim_{t\to T} \frac{\|\nabla u\|_{L^2_x}}{\|Q\|_{L^2_x}} \Big(\frac{T-t}{\ln|\ln(T-t)|} \Big)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \tag{1.41}$$

or

$$\|\nabla u(t)\|_{L^2} \geq \frac{C^*}{(T-t)\sqrt{E^G(u_0)}}, \quad E^G(u) \triangleq E(u) - \frac{1}{2}\Big(\frac{\mathrm{Im}(\int \partial_x u\bar{u})}{\|u\|_{L^2_x}} \Big)^2. \tag{1.42}$$

(ii) **Description of the singularity:** Assume $u(t)$ blows up in finite time, then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$ and an asymptotic profile $u^* \in L^2(\mathbb{R}^d)$ such that

$$u(t) - \frac{1}{\lambda(t)^{d/2}} Q\Big(\frac{x - x(t)}{\lambda(t)}\Big) e^{i\gamma(t)} \to u^* \quad \text{in} \quad L^2(\mathbb{R}^d) \quad \text{as} \quad t \to T. \tag{1.43}$$

Moreover, the blow up point is finite in the sense that

$$x(t) \to x(T) \in \mathbb{R}^d, \quad \text{as} \quad t \to T.$$

Moreover, assume $u(t)$ satisfies (1.41), $x(T)$ be its blow up point. Set

$$\lambda_0(t) = \sqrt{2\pi} \sqrt{\frac{T - t}{\ln|\ln(T - t)|}} \tag{1.44}$$

then there exists a phase parameter $\gamma_0(t) \in \mathbb{R}$ such that:

$$u(t) - \frac{1}{\lambda_0(t)^{d/2}} Q\Big(\frac{x - x_0(T)}{\lambda(t)}\Big) e^{i\gamma_0(t)} \to u^* \quad \text{in} \quad L^2(\mathbb{R}^d) \quad \text{as} \quad t \to T. \tag{1.45}$$

(iii) **Universality of blow up profile in $\dot{H}^1$:** Assume that $u(t)$ blows up in finite time with (1.41), then there exist parameters $\lambda_0(t) = \frac{\|\nabla Q\|_{L_x^2}}{\|\nabla u(t)\|_{L_x^2}}$, $x_0(t) \in \mathbb{R}^d$ and $\gamma_0(t) \in \mathbb{R}$ such that

$$e^{i\gamma_0(t)}\lambda_0(t)^{\frac{d}{2}}u(t, \lambda_0(t)x + x_0(t)) \to Q \quad \text{in} \quad \dot{H}^1, \quad \text{as} \quad t \to T. \tag{1.46}$$

If $u(t)$ satisfies (1.42), then (1.46) holds on a sequence $t_n \to T$.

(iv) **Sufficient condition for log-log blow-up:** if $E^G(u_0) < 0$, then $u(t)$ blows up in finite time with the log-log speed (1.41). More generally, the set of initial data $u_0 \in \mathcal{B}_{\alpha^*}$ such that the corresponding solution $u(t)$ to (1.32) blows up in finite time $0 < T < +\infty$ with the log-log speed (1.41) is open in H^1.

(iv) **Asymptotic of u^* on the singularity:** assume $T < +\infty$; if $u(t)$ satisfies (1.41), then for $R > 0$ small,

$$\frac{1}{C^*(\ln|\ln(R)|)^2} \leq \int_{|x-x(T)|<R} |u^*(x)|^2 \leq \frac{C^*}{(\ln|\ln(R)|)^2} \tag{1.47}$$

which implies $u^* \notin H^1$ and $u^* \notin L^p$ with $p > 2$. If $u(t)$ satisfies (1.42), then

$$\int_{|x-x(T)|\leq R} |u^*(x)|^2 \leq C^* E_0 R^2, \text{ and } u^* \in H^1. \tag{1.48}$$

Now existence of log-log solutions in $E_0(M, \alpha^*(M))$ for arbitrarily large $M > 0$ follows from the following stability result.

Theorem 1.17 (Stability of the log-log dynamic, Merle-Raphael[164])

Let $d = 1$ or $d \geq 2$ assuming spectral property holds true. Let $M > \int Q^2$, then

1. H^1 stability of the log-log regime: the set of initial data $u_0 \in E_0(M, \alpha^*(M))$ such that $u(t)$ satisfies log-log law (1.41) is open in H^1.
2. Stability of the log-log regime under large H^1 deformation: let $u_0 \in E_0(M, \alpha^*(M))$ such that (1.41) holds. Then for all $v_0 \in H^1$, there exists a time $t(v_0)$ such that $\forall\, \tilde{t}_0 \in [t(v_0), T_u)$, the solution $w(t)$ to (1.32) with initial data $w(0) = u(\tilde{t}_0) + v_0$ satisfies:

$$w(0) \in E_0\Big(M + \|v_0\|_{L^2}, \alpha^*(M + \|v_0\|_{L^2})\Big) \text{ and } w(t) \text{ satisfies log-log law (1.41).}$$

soliton resolution conjecture for mass-critical focusing NLS

Conjecture 2 (soliton resolution conjecture for mass-critical focusing NLS)

Let $u(t) \in H^1$ be a solution to (1.32) which blows up in finite time $0 < T < +\infty$. Then there exist $\{x_i\}_{1\leq i\leq L} \subset \mathbb{R}^d$ with $L \leq \frac{\|u_0\|^2_{L^2_x}}{\|Q\|^2_{L^2_x}}$ and $u^* \in L^2$ such that: $\forall\, R > 0$

$$u(t) \to u^* \quad \text{in} \quad L^2\Big(\mathbb{R}^d - \bigcup_{1\leq i\leq L} B(x_i, R)\Big),$$

and

$$|u(t)|^2 \rightharpoonup \sum_{i=1}^{L} m_i \delta_{x=x_i} + |u^*|^2 \quad \text{with } m_i \in \Big[\|Q\|^2_{L^2_x}, +\infty\Big).$$

- We will see in Section 3 that such conjecture does not hold for mass-supercritical.

Mass concentration for mass-critical focusing NLS

- Merle and Tsutsumi [171] proved that any blowup solution must concentrate at least the mass of the ground state at the blowup time; more precisely,

Theorem 1.18 (Mass concentration, Merle, Tsutsumi [171])

Let $u_0 \in H^1(\mathbb{R}^d)$ and u be the solution to (1.32) which blows up in finite time $0 < T < +\infty$. Then, there exists $x(t) \in \mathbb{R}^d$ such that

$$\lim_{t\to T} \int_{|x-x(t)|\le R} |u(t,x)|^2\, dx \ge \int_{\mathbb{R}^d} |Q|^2\, dx, \quad \forall\, R > 0. \tag{1.49}$$

- The first blowup result for general L_x^2 initial data belongs to Bourgain [15], where he obtained the following parabolic concentration of mass at the blowup time:

$$\lim_{t\to T} \sup_{\substack{\mathrm{cube}(I)\subset\mathbb{R}^2 \\ \mathrm{side}(I)<(T-t)^{\frac12}}} \left(\int_I |u(t,x)|^2\, dx\right)^{\frac12} \ge c(\|u_0\|_{L_x^2(\mathbb{R}^2)}) > 0, \tag{1.50}$$

with $c(\|u_0\|_{L^2_x(\mathbb{R}^2)})$ being a small constant depending on the mass of the initial data. This result was extended to dimension $d = 1$ by Keraani [101], and to dimensions $d \geq 3$ by Begout and Vargas [5].

Conjecture 3 (Mass concentration)

Let $u_0 \in L^2(\mathbb{R}^d)$. Assume the solution u to (1.32) blows up in finite time $0 < T < +\infty$. Then,

$$\lim_{t \to T} \sup_{\substack{\mathrm{cube}(I) \subset \mathbb{R}^2 \\ \mathrm{side}(I) < (T-t)^{\frac{1}{2}}}} \left(\int_I |u(t,x)|^2 \, dx \right)^{\frac{1}{2}} \geq \int_{\mathbb{R}^d} |Q|^2 \, dx. \tag{1.51}$$

Theorem 1.19 ($H^s(\mathbb{R}^d)$-initial data, Visan, Zhang[240])

Assume $d \geq 3$ and $s > s_0(d)$ with

$$s_0(d) := \begin{cases} \frac{1+\sqrt{13}}{5}, & \text{for} \quad d = 3, \\ \frac{8-d+\sqrt{9d^2+64d+64}}{2(d+10)}, & \text{for} \quad d \geq 4. \end{cases} \tag{1.52}$$

Let $u_0 \in H^s(\mathbb{R}^d)$ such that the corresponding solution u to (1.32) blows up at time $0 < T^* < \infty$. Let $\alpha(t) > 0$ be such that

$$\lim_{t \nearrow T^*} \frac{(T^* - t)^{\frac{s}{2}}}{\alpha(t)} = 0.$$

Then, there exists $x(t) \in \mathbb{R}^d$ such that

$$\limsup_{t \nearrow T^*} \int_{|x-x(t)| \leq \alpha(t)} |u(x,t)|^2 dx \geq \int_{\mathbb{R}^d} Q^2 dx.$$

Energy subcritical NLS

Consider

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^{p-1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{2.1}$$

$u(t,x): \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\mu = \pm 1$ corresponding to defocusing/focusing. The equation (2.1) is $\dot{H}^{s_c}$-critical with $s_c = \frac{d}{2} - \frac{2}{p-1}$, $s_c < 1$.

Defocusing case in energy space $H^1(\mathbb{R}^d)$

- Ginibre-Velo [81] proved the scattering in spatial dimension $d \geq 3$ by making use of **the almost finite propagation speed**

$$\int_{|x|\geq a} |u(t,x)|^2 \, dx \leq \int \min\Big(\frac{|x|}{a}, 1\Big)|u(t_0)|^2 \, dx + \frac{C}{a} \cdot |t - t_0|$$

for large spatial scale and **the classical Morawetz inequality** in Lin-Strauss[135]

$$\iint_{\mathbb{R}\times\mathbb{R}^d} \frac{|u(t,x)|^{p+1}}{|x|} \, dtdx \lesssim \|u\|^2_{L_t^\infty \dot{H}^{\frac{1}{2}}} \lesssim C\Big(M(u_0), E(u_0)\Big) \tag{2.2}$$

for small spatial scale to show

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{L_x^{p+1}} = 0.$$

1. For classical argument of proof, please see Miao's lecture on "Scattering Theory of the Critical Nonlinear Dispersive Equations"
2. One easily give a simple proof by usingthe interaction Morawetz estimate, see Tao-Visan-Zhang[231].

Remark on the Interactive Morawetz estimates

We can use the interaction Morawetz estimates to give a new proof of the scattering for the subcritical Schrödinger equations.

$$M^y(t) = 2\mathrm{Im}\int_{\mathbb{R}^d}\Big(\frac{x-y}{|x-y|}\cdot\nabla u\Big)\bar{u}dx, \quad \text{Morawetz Action at } y.$$

$$M^{interact}(t) = \int_{\mathbb{R}^d}|u(t,y)|^2 M^y(t)dy, \quad \text{Interaction Morawetz Potential.}$$

$$\int_I\int_{\mathbb{R}^3}|u(t,x)|^4\, dxdt \lesssim \|u(0)\|_{L^2}^2\Big(\sup_{t\in I}\|u(t)\|_{\dot{H}^{1/2}}\Big)^2, \quad \text{Interaction Morawetz.}$$

Case-(I) $13/5 \le p < 5$. Strichartz estimates and interpolation imply

$$\|u(t)\|_{L^{10}_{x,t}(\mathbb{R}^3\times\mathbb{R})} + \|\nabla u(t)\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times\mathbb{R})} + \|u(t)\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times\mathbb{R})}$$
$$\lesssim \|\varphi\|_{H^1} + \|u\|^{\frac{2(5-p)}{3}}_{L^4_{x,t}(\mathbb{R}^3\times\mathbb{R})}\Big[\|u\|_{L^{10}_{x,t}(\mathbb{R}^3\times\mathbb{R})} + \|\nabla u(t)\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times\mathbb{R})} + \|u(t)\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times\mathbb{R})}\Big]^{\frac{5p-10}{3}}.$$

Case-(II) $7/3 < p < 13/5$.

$$\|u(t)\|_{L^{10}_{x,t}(\mathbb{R}^3\times\mathbb{R})} + \|\nabla u(t)\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times\mathbb{R})} + \|u(t)\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times\mathbb{R})}$$

$$\lesssim \|\varphi\|_{H^1} + \|u\|^{6p-14}_{L^4_{x,t}(\mathbb{R}^3\times\mathbb{R})}\Big[\|u\|_{L^{10}_{x,t}(\mathbb{R}^3\times\mathbb{R})} + \|u(t)\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times\mathbb{R})} + \|\nabla u(t)\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times\mathbb{R})}\Big]^{14-5p}.$$

For the case $d \geq 4$, we have the following interactive Morawetz estimate

$$\Big\||\nabla|^{\frac{3-d}{2}}(|u|^2)\Big\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \cong \int_I \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}\,dxdydt$$
$$\lesssim \|u(t)\|^3_{L^\infty L^2_x(\mathbb{R}^d)} \|\nabla u(t)\|_{L^\infty L^2_x(\mathbb{R}^d)} \lesssim_u 1,$$

Note that

$$\Big\||\nabla|^{\frac{3-d}{4}} u\Big\|^2_{L^4_{t,x}(I\times\mathbb{R}^d)} \leq \Big\||\nabla|^{\frac{3-d}{2}}(|u|^2)\Big\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim_u 1.$$

interpolate with $u \in L^\infty(I; \dot{H}^1)$ yields that

$$\|u\|_{L^{d+1}(I; L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d))} \lesssim_u 1.$$

Critical norm conjecture

Inspired by the global well-posedness results for the mass- and energy-critical cases, one is led to the following conjecture.

Conjecture 4 (Critical norm conjecture)

Let $s_c \geq 0$ and $\mu = 1$. Suppose $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a maximal-lifespan solution to (2.1) such that

$$u \in L_t^\infty \dot{H}_x^{s_c}(I \times \mathbb{R}^d). \tag{2.3}$$

Then u is global and scatters in the sense that there exist unique $u_\pm \in \dot{H}_x^{s_c}(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} u_\pm\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} = 0.$$

The first result in this direction was due to Kenig and Merle [99], who treated the cubic problem in three dimensions.

Critical norm conjecture

	$s_c = \frac{1}{2}$	$0 < s_c < 1$	$1 < s_c \leq \frac{3}{2}$	$s_c > \frac{3}{2}$
$d = 3$	KM[99]	Murphy [191]	Murphy [191] rad	
$d = 4$	Murphy [190]	Murphy [189]	DMMZ [56, 180]	LZ [137]rad
$d \geq 5$	Murphy [190]	Murphy [189]	Killip-Visan[108]	KV[108]
$d = 1, 2$		Open		

Conjecture 5 (Low regularity conjecture)

Let $s_c \geq 0$, $\mu = 1$ and $u_0 \in \dot{H}^{s_c}$. Then u is global and scatters in the sense that there exist unique $u_\pm \in \dot{H}_x^{s_c}(\mathbb{R}^d)$ such that $\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} = 0$.

- The interaction Morawetz inequality plays also an important role in the study of a low regularity problem. Where we ask what is the minimal s to ensure that problem (2.1) has either a local solution or a global solution for which the scattering hold?

$d = 3$ cubic

$s > \frac{11}{13}$	$s > \frac{5}{6}$	$s > \frac{5}{7}$	$s > \frac{2}{3}$	$s > \frac{2}{5}$	$s > \frac{1}{2}$
Bourgain [14]	I-term [40]	Dodson[49]	Su [221]	CGT [31]	Dodson [50] radial

- Such a problem was first considered by Cazenave and Weissler [27], who proved that problem (2.1) is locally well posed in $H^s(\mathbb{R}^d)$ with $s \geq \max\{0, s_c\}$ and globally well posed together with scattering for small data in $\dot{H}^{s_c}(\mathbb{R}^d)$ with $s_c \geq 0$. They used Strichartz estimates in the framework of Besov spaces. On the other hand, since the lifespan of local solutions depend only on the H^s-norm of the initial data for $s > \max\{0, s_c\}$, one can easily obtain the global well-posedness for (2.1) in two special cases: the mass subcritical case ($p < \frac{4}{d}$) for $L^2_x(\mathbb{R}^d)$-initial data and the energy-subcritical case (for $p < \frac{4}{d-2}$,if $d \geq 3$ or for $p < +\infty$ if $d \in \{1, 2\}$) for $H^1_x(\mathbb{R}^d)$-initial data by using the conservation of mass and energy respectively.

- This leaves the open problem on global well-posedness in $H^s(\mathbb{R}^d)$ in the intermediate regime $0 \leq s_c \leq s < 1$. The first progress on this direction came from the Bourgain 'Fourier truncation method [14] where refinements of Strichartz' inequality [15], high-low frequency decompositions and perturbation methods were used to show that problem (2.1) with $p = 3$ is globally wellposed in $H^s(\mathbb{R}^3)$ with $s > \frac{11}{13}$ such that
$$u(t) - e^{it\Delta}u_0 \in H^1(\mathbb{R}^3). \tag{2.4}$$
- This leads to the I-method which was derived by Keel and Tao in the study of wave maps [97]. Subsequently, I-team developed the I-method to treat many low regularity problems including the nonlinear Schrödinger equations with derivatives, the one dimensional quintic NLS, and the cubic NLS in two and three dimensions[34, 35, 36, 37, 38, 39]. Compared with the result in [14], I-team also obtained the **scattering** in $H^s(\mathbb{R}^3)$ with $s > \frac{5}{6}$ by using the I-method and the interaction Morawetz estimate in [40]. $\cdots\cdots$

Focusing case

- Consider the subcritical focusing NLS equations

$$\begin{cases} (i\partial_t + \Delta)u = \mu|u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{2.5}$$

Let Q be the ground state of the elliptic equation

$$\Delta Q - Q + |Q|^{p-1}Q = 0. \tag{2.6}$$

Then the soliton solution $e^{it}Q$ is global but does not scatter.

- There are various ways to construct solutions to (2.6), the simplest one being to look for radial solutions via a shooting method Berestycki-Lions-Peletier[9].
- The exact structure of the set of solutions to (2.6) is not known in dimension $d \geq 2$. An important rigidity property however which combines nonlinear elliptic techniques and ODE techniques is the uniqueness of the nonnegative solution to (2.6).

Existence of solitary waves

Proposition 2.1 (Existence of solitary waves, Berestycki-Lions-Peletier[9])

(i) For $d = 1$, all solutions to (2.6) are translates of

$$Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{(p-1)x}{2}\right)}\right)^{p-1}. \tag{2.7}$$

(ii) For $d \geq 2$, there exist a sequence of radial solutions $\{Q_n\}_{n\geq 0}$ with increasing L^2 norm such that Q_n vanishes n times on $\mathbb{R}^d$.

Theorem 2.2 (Uniqueness of the ground state)

All solutions to

$$\Delta Q - Q + |Q|^{p-1}Q = 0,\ Q \in H^1(\mathbb{R}^d),\ Q(x) > 0 \tag{2.8}$$

are a translate of an exponentially decreasing C^2 radial profile $Q(r)$ (see Gidas-Ni-Nirenberg[76]) which is the unique nonnegative radially symmetric solution to (2.6) (see Kwong[123]). Q is the so called ground state solution.

- Let us now observe that we may let the full group of symmetries of (2.5) act on the solitary wave $u(t,x) = e^{it}Q$ to get a $2d+2$ parameters family of solitary waves: for $(\lambda_0, x_0, \gamma_0, \beta) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$u(t,x) = \lambda_0^{\frac{2}{p-1}} Q\Big(\lambda_0(x+x_0) - \lambda_0^2\beta t\Big)e^{i\lambda_0^2 t}e^{i\frac{\beta}{2}\cdot(\lambda_0(x+x_0)-\lambda_0^2\beta t)}.$$

- These waves are moving according to the free Galilean motion and oscillating at a phase related to their size: the larger the λ_0, the wilder the oscillations in time. An explicit computation reveals that the solitary wave can be made arbitrarily small in H^1 in the subcritical regime $s_c < 0$ only. This corresponds to the **orbital stability of the ground states in the mass subcritical case.**

Below threshold of ground state: $M(u_0)E(u_0) < M(Q)E(Q)$

By compactness concentration, one can prove the scattering/blowup dichotomy as:

Theorem 2.3 (Scattering/blowup dichotomy, Duyckaerts-Holmer-Roudenko[61, 91])

Let $p = d = 3$. Let $u_0 \in H_x^1(\mathbb{R}^3)$ satisfy $M(u_0)E(u_0) < M(Q)E(Q)$.

(i) If $\|u_0\|_{L_x^2}\|u_0\|_{\dot{H}_x^1} < \|Q\|_{L^2}\|Q\|_{\dot{H}^1}$, then the solution to (2.5) with initial data u_0 is global and scatters.

(ii) If $\|u_0\|_{L_x^2}\|u_0\|_{\dot{H}_x^1} > \|Q\|_{L^2}\|Q\|_{\dot{H}^1}$ and u_0 is radial or $xu_0 \in L_x^2(\mathbb{R}^3)$, then the solution to (2.5) with initial data u_0 blows up in finite time in both time directions.

If $\psi \in H_x^1(\mathbb{R}^3)$ satisfies $\frac{1}{2}\|\psi\|_{L_x^2}^2\|\psi\|_{\dot{H}_x^1}^2 < M(Q)E(Q)$, then there exists a global solution to (2.5) that scatters to ψ forward in time. The analogous statement holds backward in time.

- Theorem 2.3 holds for general $s_0 \in (0,1)$ under the assumption $M(u_0)^{s_c}E(u_0)^{1-s_c} < M(Q)^{s_c}E(Q)^{1-s_c}$.
- Dodson and Murphy [57, 58] give another simple proof that avoids the use of concentration compactness, by using the radial Sobolev embedding and a virial/Morawetz estimate.

The condition $\|u_0\|_{L^2_x}\|u_0\|_{\dot H^1_x} < \|Q\|_{L^2}\|Q\|_{\dot H^1} \iff \|u_0\|_{L^2_x}\|u_0\|^2_{L^4} < \|Q\|_{L^2}\|Q\|^2_{L^4}$. Indeed,

Lemma 2.4 (T. Duyckaerts, S. Roudenko[66])

Let f be in H^1, $0 < s_c < 1$. Then

$$\left(\int_{\mathbb{R}^d} |\nabla f|^2 dx\right)^{s_c} M[f]^{1-s_c} < \left(\int_{\mathbb{R}^d} |\nabla Q|^2 dx\right)^{s_c} M[Q]^{1-s_c} \Longrightarrow$$

$$\left(\int_{\mathbb{R}^d} |f|^{p+1} dx\right)^{s_c} M[f]^{1-s_c} < \left(\int -\mathbb{R}^d |Q|^{p+1} dx\right)^{s_c} M[Q]^{1-s_c}. \quad (2.9)$$

Assume furthermore that

$$M[f]^{\frac{1-s_c}{s_c}} E[f] \leq M[Q]^{\frac{1-s_c}{s_c}} E[Q]. \quad (2.10)$$

Then the reverse implication to (2.9) holds, and we obtain

$$\left(\int_{\mathbb{R}^d} |\nabla f|^2 dx\right)^{s_c} M[f]^{1-s_c} < \left(\int_{\mathbb{R}^d} |\nabla Q|^2 dx\right)^{s_c} M[Q]^{1-s_c}$$
$$\iff \left(\int_{\mathbb{R}^d} |f|^{p+1} dx\right)^{s_c} M[f]^{1-s_c} < \left(\int_{\mathbb{R}^d} |Q|^{p+1} dx\right)^{s_c} M[Q]^{1-s_c}. \quad (2.11)$$

Moreover, (2.11) also holds with non-strict inequalities (in the case of equality, f is equal to Q up to space translation, scaling and phase.)

Theorem 2.5 (T. Duyckaerts, S. Roudenko[66])

Let u be a solution of (2.5), and assume that $T_+(u) = +\infty$ and

$$\limsup_{t\to+\infty} \left(\int_{\mathbb{R}^d} |u(t)|^{p+1} dx\right)^{s_c} M[u]^{1-s_c} < \left(\int_{\mathbb{R}^d} |Q|^{p+1} dx\right)^{s_c} M[Q]^{1-s_c}.$$

Then u scatters forward in time in H^1.

Threshold solutions of ground state: $M(u_0)E(u_0) = M(Q)E(Q)$

Existence of special solutions

Theorem 2.6 (Existence of special solutions besides $e^{it}Q$ at the critical mass-energy threshold, Duyckaerts-Roudenko[65])

Let $p = d = 3$. There exist two radial solutions Q^+ and Q^- of (2.5) with initial conditions $Q_0^\pm$ such that $Q_0^\pm \in \cap_{s\in\mathbb{R}} H^s(\mathbb{R}^3)$ and

1. $M[Q^+] = M[Q^-] = M[Q]$, $E[Q^+] = E[Q^-] = E[Q]$, $[0, +\infty)$ is in the (time) domain of definition of $Q^\pm$ and there exists $e_0 > 0$ such that
$$\forall t \geq 0, \quad \left\|Q^\pm(t) - e^{it}Q\right\|_{H^1} \leq Ce^{-e_0 t},$$
2. $\|\nabla Q_0^-\|_2 < \|\nabla Q\|_2$, Q^- is globally defined and scatters for negative time,
3. $\|\nabla Q_0^+\|_2 > \|\nabla Q\|_2$, and the negative time of existence of Q^+ is finite.

Classification of solution

Theorem 2.7 (Classification of solution, T. Duyckaerts, S. Roudenko [65])

Let u be a solution of (2.5) satisfying $M(u_0)E(u_0) = M(Q)E(Q)$.

1. If $\|\nabla u_0\|_2 \|u_0\|_2 < \|\nabla Q\|_2 \|Q\|_2$, then either u scatters or $u = Q^-$ up to the symmetries.
2. If $\|\nabla u_0\|_2 \|u_0\|_2 = \|\nabla Q\|_2 \|Q\|_2$, then $u = e^{it} Q$ up to the symmetries.
3. If $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$ and u_0 is radial or of finite variance, then either the interval of existence of u is of finite length or $u = Q^+$ up to the symmetries.

Beyond threshold solutions of ground state: $M(u_0)E(u_0) > M(Q)E(Q)$

Beyond threshold solutions of ground state

In [192], Nakanishi and Schlag described the global dynamics of H^1 solutions slightly above the mass-energy threshold, $\|u_0\|_{L^2_x}\|u_0\|_{\dot H^1_x} < (1+\epsilon)\|Q\|_{L^2}\|Q\|_{\dot H^1}$. Note that, in [66], Duyckaerts and Roudenko can describe solutions which are not necessarily ϵ-close to the threshold. Define the variance as

$$V(t) = \int_{\mathbb{R}^d} |x|^2 |u(x,t)|^2 \, dx. \tag{2.12}$$

Assuming finite variance $V(0) < \infty$, the following virial identities hold:

$$V_t(t) = 4\mathrm{Im} \int_{\mathbb{R}^d} x \cdot \nabla u(x,t)\, \overline{u}(x,t)\, dx, \quad \text{and} \tag{2.13}$$

$$V_{tt}(t) = 8 \int_{\mathbb{R}^d} |\nabla u(t)|^2 dx - \frac{4d(p-1)}{p+1} \int_{\mathbb{R}^d} |u(t)|^{p+1} dx \tag{2.14}$$

$$\equiv 4d(p-1)\, E[u] - 4(p-1)s_c\, \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}. \tag{2.15}$$

Theorem 2.8

Let u be a solution of (2.5). Assume $V(0) < \infty$, $u_0 \in H^1(\mathbb{R}^d)$, and

$$\mathcal{ME}[u]\left(1 - \frac{(V_t(0))^2}{32\,E[u]\,V(0)}\right) \le 1,\ \mathcal{ME}[u] := \frac{M[u]^{\frac{1-s_c}{s_c}}E[u]}{M[Q]^{\frac{1-s_c}{s_c}}E[Q]}. \tag{2.16}$$

Part 1 (Blow up) If

$$M[u_0]^{1-s_c}\left(\int |u_0|^{p+1}\right)^{s_c} > M[Q]^{1-s_c}\left(\int |Q|^{p+1}\right)^{s_c} \tag{2.17}$$

and

$$V_t(0) \le 0, \tag{2.18}$$

then $u(t)$ blows-up in finite positive time, $T_+(u) < \infty$.

Part 2 (Boundedness and scattering) If

$$M[u_0]^{1-s_c}\left(\int |u_0|^{p+1}\right)^{s_c} < M[Q]^{1-s_c}\left(\int |Q|^{p+1}\right)^{s_c} \tag{2.19}$$

and

$$V_t(0) \geq 0, \tag{2.20}$$

then $T_+ = +\infty$, u scatters forward in time in H^1 and

$$\limsup_{t\to+\infty} \; M[u_0]^{1-s_c}\left(\int |u(t)|^{p+1}\right)^{s_c} < M[Q]^{1-s_c}\left(\int |Q|^{p+1}\right)^{s_c}. \tag{2.21}$$

Blowup rate

Blowup rate

- In the setting of arbitrarily large initial data, little is known regarding the description of the singularity formation. This is mainly a consequence of the fact that the virial blow up argument does not provide any insight into the blow up dynamics.

- More generally, the a priori control of the blow up speed $\|\nabla u\|_{L^2}$ which plays a fundamental role for the classification of blow up dynamics for example for the heat or the wave equation, is poorly understood. However a general lower bound on the blow up rate holds as a very simple consequence of the scaling invariance of the problem:

Scaling lower bound on blow up rate

Theorem 2.9 (Scaling lower bound on blow up rate, Cazenave[24])

Let $d \geq 1$, $0 \leq s_c < 1$. Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (2.5) blows up in finite time $0 < T < +\infty$, then there holds:

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^d)} \geq \frac{C(u_0)}{|T-t|^{\frac{1-s_c}{2}}}, \quad \forall\, t \in [0, T). \tag{2.22}$$

- One can ask for the sharpness of the bound (2.22), or equivalently for the existence of self similar solutions in the energy space, i.e. solutions which blow according to the scaling law

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^d)} \sim \frac{C(u_0)}{|T-t|^{\frac{1-s_c}{2}}}. \tag{2.23}$$

- For $s_c = 0$, it is an important open problem, see Bourgain [17]. It is however proved in Merle-Raphael, [197, 162] that the lower bound (2.22) is not sharp for data near the ground state in connection with the log log law, see Theorem 1.16.

On the contrary, for $s_c > 0$, a stable self-similar blow up regime in the sense of (2.23) is observed numerically, C. Sulem, P.-L. Sulem[222], and a rigorous derivation of these solutions is obtained in Merle-Raphael-Szeftel[168] for slightly super critical problems:

Theorem 2.10 (Existence and stability of self similar solutions, Merle-Raphael-Szeftel [168])

Let $1 \leq d \leq 5$ and $0 < s_c \ll 1$. Then there exists an open set of initial data $u_0 \in H^1$ such that the corresponding solution to (2.5) blows up with in finite time $T = T(u_0) < +\infty$ with the self similar speed:

$$\|\nabla u(t)\|_{L^2_x(\mathbb{R}^d)} \sim \frac{C(u_0)}{|T-t|^{\frac{1-s_c}{2}}}. \tag{2.24}$$

- The extension of this result to the full critical range $s_c < 1$ is an important **open problem**, in particular to address the physical case $d = p = 3, s_c = \frac{1}{2}$, but is confronted to the construction and the understanding of the stationary self similar profiles which is poorly understood, see [168] for a further discussion.

Blow up of the critical norm

Theorem 2.11 (Blow up of the critical norm, Merle-Raphael[165])

Let $d \geq 2$, $0 < s_c < 1$, $p < 5$. There exists a universal constant $\gamma = \gamma(d,p) > 0$ such that the following holds true. Let $u_0 \in H^1$ with radial symmetry and assume that the corresponding solution to (2.5) blows up in finite time $T < +\infty$. Then there holds the lower bound for t close enough to T:

$$\|u(t)\|_{\dot{H}^{s_c}} \geq |\log(T-t)|^{\gamma(d,p)}. \tag{2.25}$$

Theorem 2.12 (General upper bound on blow up rate, Merle[155])

Let $0 < s_c < 1$ and $u_0 \in \Sigma$ such that the corresponding solution to (2.5) blows up in finite time $0 < T < +\infty$, then:

$$\int_0^T (T-\tau)\left\|\nabla u(\tau)\right\|^2_{L^2_x(\mathbb{R}^d)}\, d\tau < +\infty. \tag{2.26}$$

Theorem 2.13 (Sharp upper bound for radial data, F. Merle-Raphael-Szeftel[170])

Let $d \geq 2,\ 0 < s_c < 1,\ p < 5$. Let the interpolation number

$$\alpha = \frac{5-p}{(p-1)(d-1)}. \tag{2.27}$$

Let $u_0 \in H^1$ with radial symmetry and assume that the corresponding solution $u \in C([0,T), H^1)$ blows up in finite time $T < +\infty$. Then there holds the space time upper bound:

$$\int_t^T (T-\tau)\left\|\nabla u(\tau)\right\|^2_{L^2_x(\mathbb{R}^d)}\, d\tau \leq C(u_0, T)(T-t)^{\frac{2\alpha}{1+\alpha}}. \tag{2.28}$$

Theorem 2.13 shows that there exists a sequence $t_n \to T$ such that

$$\left\|\nabla u(t_n)\right\|_{L^2_x(\mathbb{R}^d)} \lesssim \frac{1}{(T-t_n)^{\frac{1}{1+\alpha}}}.$$

Note that it would be very interesting to obtain the pointwise bound for all times.

Energy critical NLS

Energy critical NLS

Consider

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^{\frac{4}{d-2}} u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{2.29}$$

$u(t,x): \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\mu = \pm 1$ corresponding to defocusing/focusing.

Defocusing case: $\mu = 1$. For any $u_0 \in \dot{H}^1(\mathbb{R}^d)$, the solution to (2.29) is global and scatters.

Defocusing energy-critical

	$d = 3$	$d = 4$	$d \geq 5$
radial	Bourgain [16]	Tao [226]	Tao [226]
nonradial	I-term[39]	Ryckman-Visan [210, 239]	Visan [237, 238]

Focusing case: $\mu = -1$.

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^{\frac{4}{d-2}}u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{2.30}$$

Let W be the ground state of the elliptic equation $\Delta W + |W|^{\frac{4}{d-2}} W = 0$. An explicit solution is the stationnary solution in $\dot{H}^1$ (but in L^2 only if $d \geq 5$)

$$W := \frac{1}{\left(1 + \frac{|x|^2}{d(d-2)}\right)^{\frac{d-2}{2}}}. \tag{2.31}$$

The works of Aubin and Talenti [1, 223], give the following elliptic characterization of W

$$\forall u \in \dot{H}^1, \quad \|u\|_{L^{2^*}} \leq C_d \|u\|_{\dot{H}^1} \tag{2.32}$$

$$\|u\|_{L^{2^*}} = C_d \|u\|_{\dot{H}^1} \Longrightarrow \exists\, \lambda_0, x_0, z_0 \quad u(x) = z_0 W\left(\frac{x + x_0}{\lambda_0}\right), \tag{2.33}$$

where C_d is the best Sobolev constant in dimension d.

Below threshold of ground state $E(u_0) < E(W)$

Then, W is global but does not scatter. The first work is due to Kenig-Merle[98], where they obtain that for $E(u_0) < E(W)$, if $\|\nabla u_0\|_{L^2_x} < \|\nabla W\|_{L^2_x}$, then the solution is global and scatters; if $\|\nabla u_0\|_{L^2_x} > \|\nabla W\|_{L^2_x}$, then the solution blows up in finite time.

Focusing energy-critical

	$d = 3$	$d = 4$	$d \geq 5$
radial	Kenig-Merle [98]	Kenig-Merle [98]	
nonradial	Open!	Dodson[55]	Killip-Visan [109]

Lemma 2.14

Let f be in $\dot{H}^1$. Then

$$\|\nabla f\|_{L^2} < \|\nabla W\|_{L^2} \Longrightarrow \int |f|^{\frac{2d}{d-2}} < \int |W|^{\frac{2d}{d-2}}. \tag{2.34}$$

Assume furthermore that

$$E[f] \leq E[W]. \tag{2.35}$$

Then the reverse implication to (2.34) holds, and we obtain

$$\|\nabla f\|_{L^2} < \|\nabla W\|_{L^2} \iff \int |f|^{\frac{2d}{d-2}} < \int |W|^{\frac{2d}{d-2}}. \tag{2.36}$$

Moreover, (2.36) also holds with non-strict inequalities (in the case of equality, f is equal to W up to space translation, scaling and phase.)

Theorem 2.15 (Duyckaerts-Roudenko[66])

Let u be a solution of (2.30) with maximal time of existence $T_+(u)$, and assume

$$\limsup_{t\to T_+(u)} \int_{\mathbb{R}^N} |u(x,t)|^{\frac{2d}{d-2}}\, dx < \int |W(x)|^{\frac{2d}{d-2}}\, dx. \tag{2.37}$$

Assume furthermore that u is radial if $d = 3, 4$. Then $T_+(u) = +\infty$ and u scatters forward in time.

Remark 2.1

The assumption (2.37) is weaker than $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, due to (2.34). Try to remove the radial restriction in dimension four.

Threshold solutions of ground state $E(u_0) = E(W)$

Existence of special solutions(besides W) at threshold

Theorem 2.16 (Existence of special solutions, Duyckaerts-Merle[64])

Let $d \in \{3, 4, 5\}$. There exist radial solutions W^- and W^+ of (2.30) such that

$$E(W) = E(W^+) = E(W^-), \tag{2.38}$$

$$T_+(W^-) = T_+(W^+) = +\infty \text{ and } \lim_{t\to+\infty} W^\pm(t) = W \text{ in } \dot{H}^1, \tag{2.39}$$

$$\left\|W^-\right\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}, \quad T_-(W^-) = +\infty, \quad \|W^-\|_{S((-\infty,0])} < \infty, \tag{2.40}$$

$$\left\|W^+\right\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}, \text{ and, if } d = 5,\ T_-(W^+) < +\infty. \tag{2.41}$$

Remark 2.2

As for W, $W^+(t)$ and $W^-(t)$ belongs to L^2 if and only if $d = 5$. We still expect $T_-(W^+) < +\infty$ for $d = 3, 4$.

Theorem 2.17 (Classification of solutions, Duyckaerts-Merle [64])

Let $d \in \{3,4,5\}$. Let $u_0 \in \dot{H}^1$ radial, such that

$$E(u_0) = E(W) = 1/(NC_d^d). \tag{2.42}$$

Let u be the solution of (2.30) with initial condition u_0 and I its maximal interval of definition. Then the following holds:

1. If $\int |\nabla u_0|^2 < \int |\nabla W|^2 = \frac{1}{C_d^d}$ then $I = \mathbb{R}$. Furthermore, either $u = W^-$ up to the symmetry of the equation, or $\|u\|_{S(\mathbb{R})} < \infty$.
2. If $\int |\nabla u_0|^2 = \int |\nabla W|^2$ then $u = W$ up to the symmetry of the equation.
3. If $\int |\nabla u_0|^2 > \int |\nabla W|^2$, and $u_0 \in L^2$ then either $u = W^+$ up to the symmetry of the equation, or I is finite.

Beyond threshold solutions of ground state $E(u_0) > E(W)$

Define the variance as

$$V(t) = \int |x|^2 |u(x,t)|^2 \, dx. \tag{2.43}$$

Assuming $V(0) < \infty$ (referred to as finite variance), the following virial identities hold:

$$V_t(t) = 4\mathrm{Im} \int x \cdot \nabla u(x,t)\, \overline{u}(x,t)\, dx, \quad \text{and} \tag{2.44}$$

$$V_{tt}(t) = 8 \int |\nabla u(t)|^2 - 8 \int |u(t)|^{\frac{2d}{d-2}} \tag{2.45}$$

$$\equiv \frac{16d}{d-2} E[u] - \frac{16}{d-2} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}. \tag{2.46}$$

Theorem 2.18 (Classification of solutions, Duyckaerts-Roudenko[66])

Let u be a solution of (2.30). Assume $V(0) < \infty$, $u_0 \in H^1(\mathbb{R}^N)$, and

$$\mathcal{ME}[u]\left(1 - \frac{(V_t(0))^2}{32\,E[u]\,V(0)}\right) \leq 1,\ \mathcal{ME}[u] := \frac{\|\nabla u\|_{L^2}}{\|\nabla W\|_{L^2}}. \tag{2.47}$$

Part 1 (Blow up) If

$$\int |u_0|^{\frac{2d}{d-2}} > \int |W|^{\frac{2d}{d-2}}, \text{ and } V_t(0) \leq 0, \tag{2.48}$$

then $u(t)$ blows-up in finite positive time, $T_+(u) < \infty$.

Part 2 (Boundedness and scattering) If

$$\int |u_0|^{\frac{2d}{d-2}} < \int |W|^{\frac{2d}{d-2}}, \text{ and } V_t(0) \geq 0, \tag{2.49}$$

then

$$\limsup_{t \to T_+(u)} \int |u(t)|^{\frac{2d}{d-2}} < \int |W|^{\frac{2d}{d-2}}. \tag{2.50}$$

Furthermore, u scatters forward in time in $\dot{H}^1$ provided $d \geq 5$ or u is radial.

Existence of standing ring solutions

Theorem 2.19 (Existence and stability of a solution blowing up on a sphere in $\mathbb{R}^3$,Raphael-Szeftel[205])

Let $Q(x) = \left(\frac{3}{\cosh^2(x)}\right)^{\frac{1}{4}}$ be the ground state of the one dimension elliptic equation

$$\Delta Q - Q + Q^4 = 0.$$

There exists an open subset $\mathcal{P} \subset H^d_{\mathrm{rad}}(\mathbb{R}^3)$ such that the following holds true:

- Let $u_0 \in \mathcal{P}$, then the corresponding solution $u(t)$ to $i\partial_t u + \Delta u + |u|^4 u = 0$ blows up in finite time $0 < T < +\infty$ according to the following dynamics. There exist $\lambda(t) > 0, r(t) > 0$ and $\gamma(t) \in \mathbb{R}$ such that

$$u(t,r) - \frac{1}{\lambda(t)^{\frac{1}{2}}} Q\left(\frac{r - r(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^*(r) \quad \text{in} \quad L^2 \text{ as } t \to T. \tag{2.51}$$

Here the radius of the singular circle converges

$$r(t) \to r(T) > 0 \text{ as } t \to T \tag{2.52}$$

and

$$\lambda(t)\left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}} \to \frac{\sqrt{2\pi}}{\|Q\|_{L^2}} \text{ as } t \to T. \tag{2.53}$$

Moreover, one derivatives propagate outside the singularity:

$$\forall\ R > 0,\ u^* \in H^1\Big(|r - r(T)| > R\Big). \tag{2.54}$$

Standing ring solutions: log-log blow up solutions..

Collapsing ring solutions: polynomial blow up solutions.

Energy supercritical NLS

Existence of weak solutions Let $\Omega \subset \mathbb{R}^d$ be any open domain, and $\mu > 0,\ p > 1$.

Consider

$$\begin{cases} iu_t + \Delta u = \mu |u|^{p-1} u, & (t,x) \in \mathbb{R} \times \Omega \\ u(0,x) = u_0(x). \end{cases} \tag{2.55}$$

- If $p - 1 \leq \frac{4}{d-2}(p < +\infty$, if $d = 1,2)$, the problem (2.55) has a solution

$$u \in L^\infty(\mathbb{R}, H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega))$$

  for every $u_0 \in H_0^1(\Omega)$.

- In addition, if $\Omega = \mathbb{R}^d$, or if $d = 1$, or if $d = 2$ and $p \leq 3$, the solution is unique.
- However, **those results do not apply when** $p - 1 > \frac{4}{d-2}$.

Here, we present a result of Strauss [218](see also [217]) that applies for arbitrarily large p by using compactness method.

Theorem 2.20 (Strauss [218], or [24] Theorem 9.4.1)

Let $\eta > 0$ and $p > 1$. It follows that for every $u_0 \in V := H_0^1(\Omega) \cap L^{p+1}(\Omega)$, there exists a solution $u \in L^\infty(\mathbb{R}, V) \cap W^{1,\infty}(\mathbb{R}, V^*)$ with $V^* = H^{-1}(\Omega) \bigoplus L^{\frac{p+1}{p}}(\Omega)$ of equation (2.55) satisfies

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \text{for all } t \in \mathbb{R}; \tag{2.56}$$

$$E(u(t)) \leq E(u_0), \quad \text{for all } t \in \mathbb{R}. \tag{2.57}$$

Remark 2.3

(i) Note that, in particular, $u \in C(\mathbb{R}, V^*)$, and so u is weakly continuous $\mathbb{R} \to H_0^1(\Omega)$ and $\mathbb{R} \to L^{p+1}(\Omega)$; in particular, $u(t) \in V$ for all $t \in \mathbb{R}$. Therefore, $u(0)$ makes sense in V and $E(u(t))$ is well defined for all $t \in \mathbb{R}$.

(ii) Note that when $p - 1 \leq \frac{4}{d-2}$, then $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, therefore, $V = H_0^1(\Omega)$. Theorem 2.20 in this part follows from the above classical argument.

(iii) The proof of Theorem 2.20 does not apply in the focusing energy supercritical case.

Consider

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^{p-1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}^d \end{cases} \tag{2.58}$$

$u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\mu = \pm 1$ corresponding to defocusing/focusing. $s_c > 1$, i.e.

$$p > 1 + \frac{4}{d-2}.$$

Critical norm conjecture

	$s_c = \frac{1}{2}$	$0 < s_c < 1$	$1 < s_c \leq \frac{3}{2}$	$s_c > \frac{3}{2}$
$d = 3$	KM[99]	Murphy [191]	Murphy [191] rad	
$d = 4$	Murphy [190]	Murphy [189]	DMMZ [56, 180]	LZ [137]rad
$d \geq 5$	Murphy [190]	Murphy [189]	Killip-Visan[108]	KV[108]
$d = 1,2$		Open		

Type I Blow up

Consider

$$\begin{cases} (i\partial_t + \Delta)u = \mu |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{2.59}$$

- If $s_c > 1$, The problem (2.59) is not well-posed in H^1. To prove LWP in the critical Sobolev space $\dot{H}^{s_c}$, one needs the nonlinearity to be at least C^{s_c}, i.e. p **is an odd integer** or

$$d \leq 7 \quad \text{or} \quad p > \frac{d+2+\sqrt{d^2-4d-28}}{4} \qquad \left(\iff s_c < p \right) \tag{2.60}$$

Theorem 2.21 (The sufficient condition leading to blow-up, Duyckaerts-Roudenko[66])

Suppose that $u_0 \in H^1$ and $V(0) < \infty$. If $s_c > 1$, assume furthermore $u_0 \in \dot{H}^{s_c}$ and p is odd or (2.60). The following is a sufficient condition for blow-up in finite time for (2.59) with $s_c > 0$ and $E[u] > 0$:

$$\frac{V_t(0)}{M} < \sqrt{8ds_c}\, g\left(\frac{4}{ds_c}\, \frac{EV(0)}{M^2}\right), \quad E = E[u],\ M = M]u] \tag{2.61}$$

where

$$g(x) = \begin{cases} \sqrt{\frac{1}{kx^k} + x - \left(1+\frac{1}{k}\right)} & \text{if } 0 < x \leq 1 \\ -\sqrt{\frac{1}{kx^k} + x - \left(1+\frac{1}{k}\right)} & \text{if } x \geq 1 \end{cases} \quad \text{with} \quad k = \frac{(p-1)s_c}{2}. \tag{2.62}$$

Theorem 2.22 (The sufficient condition leading to blow-up)

Suppose that $u_0 \in H^1$ and $\|xu_0\|_{L^2} < \infty$.If $s_c > 1$, assume furthermore $u_0 \in \dot{H}^{s_c}$ and p is odd or (2.60). The following is a sufficient condition for blow-up in finite time for NLS (2.59) with $s_c > 0$ and $E[u] > 0$:

$$\frac{V_t(0)}{M} < \frac{4\sqrt{2}(M^{1-s_c}E^{s_c})^{\frac{1}{d}}}{C}\, g\left(C^2 \frac{E^{\frac{4}{d(p-1)}}V(0)}{M^{1+\frac{2(p+1)}{d(p-1)}}}\right), \tag{2.63}$$

where

$$C = \left(\frac{2(p+1)}{s_c(p-1)}\left(C_{p,d}\right)^{\frac{d(p-1)}{2}+(p+1)}\right)^{\frac{2}{d(p-1)}} \tag{2.64}$$

and $C_{p,d}$ is a sharp constant in the interpolation inequality

$$\|u\|_{L^2} \leq C_{p,N}\left(\|xu\|_{L^2}^{\frac{N(p-1)}{2}}\|u\|_{L^{p+1}}^{p+1}\right)^{1/\left(\frac{N(p-1)}{2}+(p+1)\right)}. \tag{2.65}$$

Type II Blow up

Theorem 2.23 (Type II blow up for the super critical NLS equation,Merle-Raphael-Rodnianski[167])

Let $d \geq 11$. Let $\alpha = \gamma - \frac{2}{p-1}$ and assume:

$$\begin{cases} p = 2q+1, \ \ q \in \mathbb{N}^*, \\ p > p_{JL}, \\ \mathrm{Discr} > 4 \end{cases} \tag{2.66}$$

and

$$\frac{\alpha}{2} \notin \mathbb{N}, \ \ \frac{1}{2} + \frac{1}{2}\left(\frac{d}{2} - \gamma\right) \notin \mathbb{N}, \ \ \frac{1}{2} + \frac{1}{2}\left(\frac{d}{2} - \frac{2}{p-1}\right) \notin \mathbb{N}. \tag{2.67}$$

Fix an integer

$$\ell \in \mathbb{N}^* \ \text{ with } \ \ell > \frac{\alpha}{2}, \tag{2.68}$$

and an arbitrary large Sobolev exponent

$$s^+ \in \mathbb{N}, \ \ s_+ \geq s(\ell) \to +\infty \ \text{ as } \ \ell \to +\infty.$$

Then there exists a radially symmetric initial data $u_0(r) \in H^{s_+}(\mathbb{R}^d, \mathbb{C})$ such that the corresponding solution to (2.55) blows up in finite time $0 < T < +\infty$ via concentration of the soliton profile:

$$u(t,r) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(Q+\varepsilon)\left(\frac{r}{\lambda(t)}\right)e^{i\gamma(t)} \tag{2.69}$$

with:

(i) **Blow up speed**:

$$\lambda(t) = c(u_0)(1+o_{t\uparrow T}(1))(T-t)^{\frac{\ell}{\alpha}}, \quad c(u_0) > 0; \tag{2.70}$$

(ii) **Stabilization of the phase**:

$$\gamma(t) \to \gamma(T) \in \mathbb{R} \text{ as } t \to T; \tag{2.71}$$

(iii) **Asymptotic stability above scaling**:

$$\lim_{t\uparrow T} \|\nabla^s \varepsilon(t,\cdot)\|_{L^2} = 0 \text{ for all } s_c < s \leq s_+; \tag{2.72}$$

(iv) **Boundedness below scaling**:

$$\limsup_{t\uparrow T} \|u(t)\|_{H^s} < +\infty \text{ for all } 0 \leq s < s_c; \tag{2.73}$$

(v) **Behavior of the critical norm**:

$$\|u(t)\|_{\dot{H}^{s_c}} = \left[c_\infty \sqrt{\frac{\ell}{\alpha}} + o_{t\uparrow T}(1)\right]\sqrt{|\log(T-t)|}. \tag{2.74}$$

Existence of standing ring blowup solutions

Theorem 2.24 (Existence and stability of a solution blowing up on a sphere in $\mathbb{R}^3$,Raphael-Szeftel[205])

Let $Q(x) = \left(\frac{3}{\cosh^2(x)}\right)^{\frac{1}{4}}$ be the ground state of the one dimension elliptic equation $\Delta Q - Q + Q^4 = 0$. There exists an open subset $\mathcal{P} \subset H^d_{\mathrm{rad}}(\mathbb{R}^d)$ such that the following holds true. Let $u_0 \in \mathcal{P}$, then the corresponding solution $u(t)$ to

$$i\partial_t u + \Delta u + |u|^4 u = 0,\ (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ d \geq 4$$

blows up in finite time $0 < T < +\infty$ according to the following dynamics. There exist $\lambda(t) > 0, r(t) > 0$ and $\gamma(t) \in \mathbb{R}$ such that

$$u(t,r) - \frac{1}{\lambda(t)^{\frac{1}{2}}} Q\left(\frac{r - r(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^*(r) \quad \text{in} \quad L^2 \text{ as } t \to T. \tag{2.75}$$

Here the radius of the singular circle converges

$$r(t) \to r(T) > 0 \text{ as } t \to T \tag{2.76}$$

and

$$\lambda(t)\left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}} \to \frac{\sqrt{2\pi}}{\|Q\|_{L^2}} \text{ as } t \to T. \tag{2.77}$$

Moreover, $\frac{d-1}{2}$-derivatives propagate outside the singularity:

$$\forall\, R > 0,\ u^* \in H^{\frac{d-1}{2}}\Big(|r - r(T)| > R\Big). \tag{2.78}$$

Mass subcritical NLS

Consider

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^{p-1} u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{2.79}$$

$u(t,x): \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\mu = \pm 1$ corresponding to defocusing/focusing.

- When $s_c < 0$, the solution $u(t,x)$ to (2.86) with $u_0 \in H^1(\mathbb{R}^d)$ is global in both defocusing and focusing case.

Theorem 2.25 (Strauss index)

Assume that $d \geq 1$ and $1 < p \leq 1 + \frac{2}{d}$. Let u be a nontrivial, smooth solution to (2.86), and asymptotical free, i.e. $\exists\, v_\pm \in L^2(\mathbb{R}^n)$ s.t.

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}v_\pm\|_{L^2_x(\mathbb{R}^n)} = 0. \tag{2.80}$$

Then, $v_\pm \equiv 0$. Combining this with mass conservation, we obtain $u \equiv 0$.

Theorem 2.26

Let $1 < p \leq 1 + \frac{2}{d}$. Then, there exists $u_\pm \in L^2(\mathbb{R}^d)$ of arbitrarily small mass norm such that there cannot be any strong solution u of (2.86) satisfying the following

$$\lim_{t\to\pm\infty} \left\|u(t) - e^{-it\Delta}u_\pm\right\|_{L^2(\mathbb{R}^d)} = 0. \tag{2.81}$$

Theorem 2.27 (Scattering in Σ space, Cazenave [24] Theorem 7.4.1, 7.5.11)

Let $u_0 \in \Sigma$, $a(d) = \frac{2-d+\sqrt{d^2+12d+4}}{2d}$, $\mu = 1$,

- $a(d) + 1 < p < 2^* - 1$ (see Tsutsumi[232]);
- $d = 1$ or $d \geq 3$, $p = 1 + a(d)$ (see Cazenave-Weissler [28]).

Then, the solution to (2.86) is global and scatters.

Remark 2.4

It is easy to check that $a(d)$ is the positive root of the polynomial $d\alpha^2 + (d-2)\alpha - 4 = 0$. We note that $\alpha \geq 0$ satisfies $d\alpha^2 + (d-2)\alpha - 4 > 0$ if and only if $\alpha > a(d)$. We also that that

$$\begin{cases} \frac{2}{d} \leq \frac{4}{d+2} < a(d) < \frac{4}{d}, & \text{if} \quad d \geq 2, \\ \frac{2}{d} < a(d) < \frac{4}{d} & \text{if} \quad d = 1. \end{cases} \tag{2.82}$$

Theorem 2.28 (Scattering in critical weight space, Killip-Masaki- Murphy-Visan[104])

Assume $\max\left(\frac{2}{d}, \frac{4}{d+2}\right) < p-1 < \frac{4}{d}$, $\mu \in \{\pm 1\}$. Let $t_0 \in [-\infty, \infty)$ and

$$u_0 \in \mathcal{F}\dot{H}^{|s_c|} := \{|x|^{|s_c|} f \in L^2(\mathbb{R}^d)\}.$$

Let $u : I_{\max} \times \mathbb{R}^d \to \mathbb{C}$ be the maximal-lifespan solution to (2.86) with initial condition

$$e^{-it_0\Delta} u(t_0) = u_0 \in \mathcal{F}\dot{H}^{|s_c|}. \tag{2.83}$$

Suppose

$$\sup_{t \in I_{\max}} \|e^{-it\Delta} u(t)\|_{\mathcal{F}\dot{H}^{|s_c|}} < \infty. \tag{2.84}$$

Then u is forward-global and scatters forward in time; that is, there exists $u_+ \in \mathcal{F}\dot{H}^{|s_c|}$ such that

$$\lim_{t \to \infty} \|e^{-it\Delta} u(t) - u_+\|_{\mathcal{F}\dot{H}^{|s_c|}} = 0. \tag{2.85}$$

Orbital stability

Mass-subcritical

Consider the mass-subcritical NLS

$$i\partial_t u + \Delta u + |u|^{p-1} u = 0, \;\; 1 < p < 1 + \frac{4}{d}. \tag{2.86}$$

Let us first observe that two trivial instabilities are given by the symmetries of the equation:

1. **Scaling instability:** $\forall\ \lambda > 0$, the solution to (2.86) with initial data $u_0(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ is $u(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}$;
2. **Galilean instability:** $\forall\ \beta > 0$, the solution to (2.86) with initial data $u_0(x) = e^{i\beta x} Q(x)$ is $u(t,x) = e^{it + \frac{\beta}{2}\cdot(x - \frac{\beta}{2}t)} Q(x - \beta t)$.

In both cases,

$$\sup_{t\in\mathbb{R}} \left| u(t,x) - e^{it} Q(x) \right| > |Q(x)|.$$

and thus **the solution does not stay uniformly close to Q**.

Orbital stability in mass subcritical

Theorem 2.29 (Orbital stability of the ground state, Cazenave and Lions[25])

Let $d \geq 1$ and $1 < p < 1 + \frac{4}{d}$. For all $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that the following holds true. Let $u_0 \in H^1$ with

$$\|u_0 - Q\|_{H^1} < \delta(\varepsilon).$$

then there exist a translation shift $x(t) \in C^0(R; R^d)$ and a phase shift $\gamma(t) \in C^0(R; R)$ such that:

$$\|u(t,x) - e^{i\gamma(t)}Q(x - x(t))\|_{H^1(\mathbb{R}^d)} < \varepsilon, \ \forall\, t \in \mathbb{R}. \tag{2.87}$$

Orbital instability in mass critical

Theorem 2.30 (Orbital instability in mass critical, Weinstein [243])

Let $\omega > 0$ and Q be the ground state. Then $u(t,x) = e^{i\omega t}Q$ is an unstable solution of (1.32) in the following sense. There exists a sequence $\{\varphi_m\} \subset H^1$ such that

$$\varphi_m \to Q \quad \text{in} \quad H^1$$

and such that the corresponding maximal solution u_m of (1.32) blows up in finite time for both sides.

One can refer to Berestycki and Cazenave [7] for the instability of mass-supercritical case.

Theorem 2.31 (Modified orbital instability in mass critical, Raphael[199])

Let $d \geq 1$. For all $\alpha^* > 0$ small enough, there exists $\delta(\alpha^*)$ with $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$ such that the following holds true. Let $u_0 \in H^1$ with

$$\int_{\mathbb{R}^d} |u_0|^2 dx \leq \int_{\mathbb{R}^d} Q^2 dx + \alpha^*, \quad E(u) \leq \alpha^* \int_{\mathbb{R}^d} |\nabla u|^2 dx, \tag{2.88}$$

and let $u(t)$ be the corresponding solution to (1.32) with lifespan time $0 < T \leq +\infty$, then there exists $(x(t), \gamma(t)) \in C^0([0, T), \mathbb{R}^d \times \mathbb{R})$ such that

$$\left\| \lambda(t)^{\frac{d}{2}} u(t, \lambda(t)x + x(t)) e^{-i\gamma(t)} - Q \right\|_{H^1} < \delta(\alpha^*), \quad \forall\, t \in [0, T), \tag{2.89}$$

with $\lambda(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u(t)\|_{L^2}}$. In other words,

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{d}{2}}} (Q + \varepsilon)\left(t, \frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)}, \ \|\varepsilon(t)\|_{H^1} \leq \delta(\alpha^*). \tag{2.90}$$

Theorem 2.32 (Non-squeezing, Illip-Visan-Zhang[113])

Let $\mu \in \{\pm 1\}$, $p = 3$, $d = 1$. Fix $z_* \in L^2(\mathbb{R})$, $l \in L^2(\mathbb{R})$ with $\|l\|_{L^2} = 1$, $\alpha \in \mathbb{C}$, $0 < r < R < \infty$ and $T > 0$. Then there exists $u_0 \in B(z_*, R)$ such that the solution u to (2.86) with $\mu = 1$ and initial data $u(0) = u_0$ satisfies

$$|\langle l, u(T)\rangle - \alpha| > r. \tag{2.91}$$

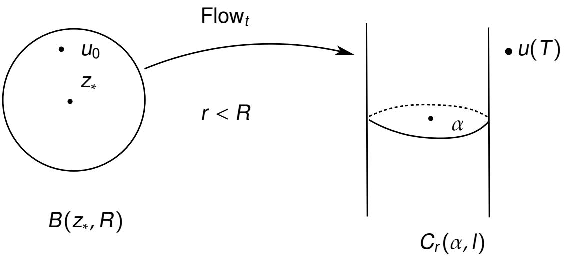

Figure:

Theorem 2.33 (Modified wave operators)

Let $\langle x\rangle^2\phi \in L^2(\mathbb{R}^2)$ and $\phi \in \dot{H}^{-\delta}$ with $1<\delta<2$, and $\left\|\langle x\rangle^2\phi\right\|_{L^2} + \|\phi\|_{\dot{H}^{-\delta}} \ll 1$. Then, there exists a unique global solution u of $i\partial_t u + \frac{1}{2}\Delta u = u^2, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^2$, such that $u\in C(\mathbb{R}^+; L^2)$ and

$$\sup_{t\geq 1} t^{\frac{\delta}{2}}\left[\|u(t)-u_p(t)\|_{L^2_x} + \|u(\tau)-u_p(\tau)\|_{L^4_{\tau,x}([t,\infty)\times\mathbb{R}^2)}\right] < \infty, \tag{2.92}$$

where $u_p(t) = \frac{1}{it}e^{\frac{i|x|^2}{2t}}\hat{\phi}\left(\frac{x}{t}\right)$. Furthermore, the modified wave operator

$$\tilde{W}_+ : \phi \to u(0)$$

is well-defined.

Some model of dispersive equations I

(1) NLS on domain: $i\partial_t u + \Delta u = |u|^{p-1}u,\ x \in M.$

(1) Wave equation

$$\partial_t^2 u - \Delta u = F(u) \tag{2.93}$$

(2) KdV equation (Korteweg-de Vries equation)

$$\partial_t u + \partial_x^3 u - 6u\partial_x u = 0. \tag{2.94}$$

Generalized KdV

$$\partial_t u + \partial_x^3 u - u^p \partial_x u = 0. \tag{2.95}$$

(3) Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u = F(u) \tag{2.96}$$

(4) Derivative NLS

$$i\partial_t u + u_{xx} + i|u|^2 u_x = 0. \tag{2.97}$$

Some model of dispersive equations II

(5) Zakharov system

$$\begin{cases} i\partial_t u + \Delta u = nu, \\ \partial_t^2 n - \Delta n = \Delta |u|^2. \end{cases} \tag{2.98}$$

(6) Nonlinear Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxxx} + 3(u^2)_{xx} = 0, \tag{2.99}$$

and modified Boussinesq equation

$$\tfrac{1}{3} u_{tt} - u_t u_{xx} - \tfrac{3}{2} u_x^2 u_{xx} + u_{xxxx} = 0. \tag{2.100}$$

(7) Schrödinger map: $\partial_t u = u \times \Delta u$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$

(8) Wave map equation: $D^\alpha \partial_\alpha u = 0$.

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