Tartar's Method

Introduced by Tartar (1977). Recall:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

$A_\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$, A 1-periodic, $\exists\ \alpha, \beta > 0$:

$$\left.\begin{aligned} \langle \xi, A(x)\xi \rangle &\geq \alpha |\xi|^2 \\ |A(x)\xi| &\leq \beta |\xi| \end{aligned}\right\} \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{R}^N.$$

For simplicity here: A symmetric.

Assumptions above imply apriori bound

$$\|u_\varepsilon\|_{H^1} \leq C \|f\|_{L^2}$$

By compact embeddings etc: Subsequence (u_ε) with

i) $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$

ii) $u_\varepsilon \longrightarrow u_0$ strongly in $L^2(\Omega)$

iii) $\xi_\varepsilon \rightharpoonup \xi_0$ weakly in $L^2(\Omega)$,

where $\xi_\varepsilon := A_\varepsilon \nabla u_\varepsilon$ "flux". One has

$$\int_\Omega \xi_\varepsilon \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega)$$

$$\Longrightarrow \int_\Omega \xi_0 \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega) \quad \text{by iii)}$$

$$\Longleftrightarrow \boxed{-\operatorname{div} \xi_0 = f}$$

Task: Identify ξ_0.

Define $w_\varepsilon^i(x) := x_i - \varepsilon \chi_i\left(\frac{x}{\varepsilon}\right)$. Then

$$w_\varepsilon^i \longrightarrow x_i \quad \text{weakly in } L^2(\Omega).$$

Also, by definition:

$$(\nabla_x w_\varepsilon^i)(x) = e_i - \nabla_y \chi_i\left(\tfrac{x}{\varepsilon}\right)$$

$\Longrightarrow \nabla_x w_\varepsilon^i$ periodic

$$\Longrightarrow \begin{aligned} \nabla_x w_\varepsilon^i &\rightharpoonup \langle e_i - \nabla_y \chi_i \rangle \\ &= e_i - \langle \nabla_y \chi_i \rangle \end{aligned} \qquad \text{weakly in } L^2(\Omega)$$

By partial integration: $\langle \nabla_y \chi_i \rangle = 0$.

Hence:

i) $w_\varepsilon^i \rightharpoonup x_i$ weakly in $H^1(\Omega)$

ii) $w_\varepsilon^i \longrightarrow \chi_i$ strongly in $L^2(\Omega)$.

(compact embedding $H^1 \hookrightarrow L^2$).

Define the 1-periodic function

$$\eta_\varepsilon^i(x) := A\left(\tfrac{x}{\varepsilon}\right) \nabla_y w_\varepsilon^i(x).$$

Then

$$\eta_\varepsilon^i \longrightarrow \left\langle A\left(e_i - \nabla_y \chi_i\right)\right\rangle \quad \text{weakly in } L^2(\Omega)$$

$$= A_0 e_i$$

<u>Claim:</u> $\displaystyle\int_\Omega \eta_\varepsilon^i \cdot \nabla v \, dx = 0 \quad \forall v \in H_0^1(\Omega)$ (*)

<u>Proof:</u>
From cell problem:

$$\int_{[0,1]^N} A \nabla_y \chi_i \cdot \nabla_y \psi \, dy = -\int_{[0,1]^N} A e_i \cdot \nabla_y \psi \, dy \quad \forall \psi \in H_\#^1([0,1]^N)$$

$$\Longleftrightarrow \int_{[0,1]^N} A\left(e_i - \nabla_y \chi_i\right) \cdot \nabla_y \psi \, dy = 0$$

Let $\varphi \in C_0^\infty(\Omega)$ and $\varphi_\varepsilon(y) := \varphi(\varepsilon y)$. One can show that the above implies

$$\int_{\mathbb{R}^N} A\left(e_i - \nabla_y \chi_i\right) \cdot \underbrace{\nabla \varphi_\varepsilon}_{= \varepsilon (\nabla\varphi)(\varepsilon x)} \, dy = 0$$

$$\overset{x = \varepsilon y}{\Longrightarrow} \int_\Omega A\left(e_i - \nabla_y \chi_i\right)\left(\tfrac{x}{\varepsilon}\right) \cdot \varepsilon (\nabla\varphi)(\varepsilon x) \frac{dx}{\varepsilon} = 0$$

$$\Longleftrightarrow \int_\Omega A_\varepsilon \nabla_y w_\varepsilon^i(x) \cdot \nabla \varphi(x) \, dx = 0$$

$$\Longleftrightarrow \int_\Omega \eta_\varepsilon^i(x) \cdot \nabla \varphi(x) \, dx = 0$$

□

Let $\varphi \in C_0^\infty(\Omega)$ and choose φw_ε^i as test function in original PDE:

(2) $\displaystyle\int \xi_\varepsilon \cdot \nabla w_\varepsilon^i \cdot \varphi \, dx + \int \xi_\varepsilon \cdot \nabla \varphi \cdot w_\varepsilon^i \, dx = \int f \varphi w_\varepsilon^i \, dx$

conv. ?? conv. ✓ conv. ✓

... chose φu_ε as test function in (1)

$$(3) \qquad \boxed{\int_\Omega \eta_\varepsilon^i \cdot \nabla u_\varepsilon \cdot \varphi \, dx} + \int_\Omega \eta_\varepsilon^i \cdot \nabla \varphi \cdot u_\varepsilon \, dx = 0 .$$

By definition:

$$\xi_\varepsilon \cdot \nabla w_\varepsilon^i = A_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon^i \overset{A \text{ symm.}}{=} \nabla u_\varepsilon \cdot A_\varepsilon \nabla w_\varepsilon^i = \nabla u_\varepsilon \cdot \eta_\varepsilon^i$$

$$\Rightarrow \int_\Omega \xi_\varepsilon \cdot \nabla \varphi \, w_\varepsilon^i \, dx - \int_\Omega \eta_\varepsilon^i \cdot \nabla \varphi \, u_\varepsilon = \int_\Omega f \varphi w_\varepsilon^i \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\int_\Omega \xi_0 \cdot \nabla \varphi \, x_i \, dx - \int_\Omega A_0 e_i \cdot \nabla \varphi \, u_0 \, dx = \int_\Omega f \varphi x_i \, dx$$

$$\Leftrightarrow \underbrace{\int_\Omega \xi_0 \cdot \nabla (x_i \cdot \varphi) \, dx}_{= \int_\Omega f x_i \varphi \, dx} - \int_\Omega \xi_0 \cdot e_i \, \varphi \, dx - \int_\Omega A_0 e_i \cdot \nabla \varphi \, u_0 \, dx = \int_\Omega f \varphi x_i \, dx$$

$$\Rightarrow \int_\Omega \xi_0 \cdot e_i \, \varphi \, dx = - \int_\Omega A_0 e_i \cdot \nabla \varphi \, u_0 \, dx$$

$$= + \int_\Omega A_0 e_i \cdot \nabla u_0 \, \varphi \, dx$$

$$= \int_\Omega A_0 \nabla u_0 \cdot e_i \, \varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$$\Rightarrow \quad \xi_0 \cdot e_i = A_0 \nabla u_0 \cdot e_i \quad \forall i$$

$$\Rightarrow \quad \xi_0 = A_0 \nabla u_0$$

On the whole: $u_\varepsilon \longrightarrow u_0$ in $H^1(\Omega)$, where u_0 solves

$$\begin{cases} -\operatorname{div}(A_0 \nabla u_0) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A_0 e_i = \langle A e_i - A \nabla \chi_i \rangle$