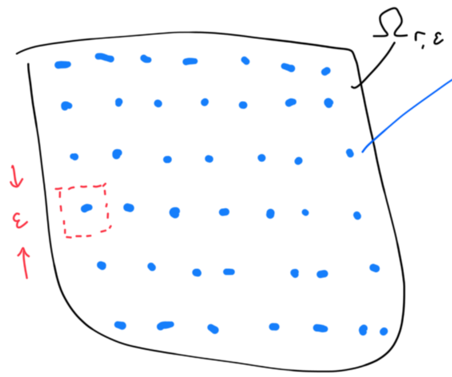
Perforated domains:

Motivation: Crushed ice:



Temperature modelled by

$$\left.\begin{aligned} \partial_t u_{r,\varepsilon} - \Delta u_{r,\varepsilon} &= f && \text{in } \Omega_{r,\varepsilon} \\ u_{r,\varepsilon} &= 0 && \text{on } \partial\Omega_{r,\varepsilon} \end{aligned}\right\}$$

Heuristically:

- If ε fixed, $r \to 0$, then in the limit

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{on } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- If $r \sim \varepsilon$ as $\varepsilon \to 0$, then total mass of ice remains constant, while surface of ice grows unboundedly

  Mass: $m \approx \frac{1}{\varepsilon^N} \cdot r^N \sim$ const.

  Surface: $\sigma \approx \frac{1}{\varepsilon^N} \cdot r^{N-1} \sim \frac{1}{\varepsilon}$

  $\rightsquigarrow \quad u_{r,\varepsilon} \longrightarrow 0 \quad$ as $\quad r \sim \varepsilon \longrightarrow 0$.

$\rightsquigarrow$ Question: What about intermediate scalings?

Let $\Omega \subset \mathbb{R}^N$ open, bounded, let $T_i^\varepsilon \subset \mathbb{R}^N$ closed for $1 \leq i \leq n(\varepsilon)$.

Define

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^{n(\varepsilon)} T_\varepsilon^i$$

and for $f \in L^2(\Omega)$ consider

$$\left.\begin{aligned} -\Delta u_\varepsilon &= f \quad \text{in } \Omega_\varepsilon \\ u_\varepsilon &\in H_0^1(\Omega_\varepsilon) \end{aligned}\right\}$$

Weak formulation: Find $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$ s.t.

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla v \, dx = \int_{\Omega_\varepsilon} f v \, dx \qquad \forall v \in H_0^1(\Omega_\varepsilon).$$

Denote $\tilde{u}_\varepsilon := \begin{cases} u_\varepsilon & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \bigcup_i T_\varepsilon^i \end{cases}$. Then

$$\|\tilde{u}_\varepsilon\|_{H^1(\Omega)}^2 \overset{\text{Poincaré}}{\leq} C \|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)}^2 = C \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 = C \int_{\Omega_\varepsilon} f u_\varepsilon \, dx$$

$$\leq C \int_\Omega f \tilde{u}_\varepsilon \, dx$$

$$\leq C \|f\|_{L^2(\Omega)} \|\tilde{u}_\varepsilon\|_{L^2(\Omega)}$$

$\Longrightarrow$ $(\tilde{u}_\varepsilon)$ bounded in $H^1(\Omega)$

$\Longrightarrow$ conv. subsequence $\tilde{u}_\varepsilon \rightharpoonup u_0$ in $H^1(\Omega)$

$\rightsquigarrow$ $\tilde{u}_\varepsilon \longrightarrow u_0$ in $L^2(\Omega)$.

$\rightsquigarrow$ Question: Can u_0 be identified?

Hypothesis: There exist functions w_ε and distribution μ s.t.

(H1) $w_\varepsilon \in H^1(\Omega)$

(H2) $w_\varepsilon = 0$ on T_ε^i, $1 \leq i \leq n(\varepsilon)$

(H3) $w_\varepsilon \rightharpoonup 1$ weakly in $H^1(\Omega)$

(H4) $\mu \in W^{-1,\infty}(\Omega) = \left(W_0^{1,1}(\Omega)\right)'$

(H5) $\left\{\begin{array}{l} \text{for every sequence } (v_\varepsilon) \text{ s.t. } v_\varepsilon = 0 \text{ on } \bigcup_i T_\varepsilon^i \\ \text{satisfying } v_\varepsilon \xrightharpoonup{H^1(\Omega)} v \text{ one has} \\ \qquad \langle -\Delta w_\varepsilon, \varphi v_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \longrightarrow \langle \mu, \varphi v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ \text{for all } \varphi \in C_0^\infty(\Omega). \end{array}\right.$

Theorem:

Under (H1)–(H5):

$$\tilde{u}_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega),$$

where u_0 solves

$$\left.\begin{array}{r} (-\Delta + \mu) u_0 = f \quad \text{in } \Omega \\ u_0 \in H_0^1(\Omega) \end{array}\right\}$$

Proof:

From above: $u_\varepsilon \rightharpoonup u_0$ w. in $H^1(\Omega)$.

Like Tartar: Let $\varphi \in C_0^\infty(\Omega)$ and use $w_\varepsilon \varphi$ as test function in weak formulation:

$$\int_{\Omega_{\cancel{\varepsilon}}} \nabla \tilde{u}_\varepsilon \nabla (w_\varepsilon \varphi)\, dx = \int_{\Omega_{\cancel{\varepsilon}}} f\, w_\varepsilon \varphi\, dx$$

$$\int_\Omega \nabla \tilde{u}_\varepsilon \cdot \nabla w_\varepsilon \, \varphi \, dx + \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi \, w_\varepsilon \, dx = \int_\Omega f \, w_\varepsilon \, \varphi \, dx$$

$$\int_\Omega \nabla \tilde{u}_\varepsilon \cdot \nabla w_\varepsilon \, \varphi \, dx \longrightarrow \, ? \qquad \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi \, w_\varepsilon \, dx \xrightarrow{(H3)} \int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx \qquad \int_\Omega f \, w_\varepsilon \, \varphi \, dx \xrightarrow{(H3)} \int_\Omega f \varphi \, dx$$

First term:

$$\int_\Omega \nabla \tilde{u}_\varepsilon \cdot \nabla w_\varepsilon \, \varphi \, dx = \int_\Omega \nabla w_\varepsilon \cdot \nabla (\tilde{u}_\varepsilon \varphi) \, dx - \underbrace{\int_\Omega \nabla w_\varepsilon \cdot \nabla \varphi \, u_\varepsilon \, dx}_{\downarrow \, 0}$$

$$= \langle -\Delta w_\varepsilon , \tilde{u}_\varepsilon \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + o(1)$$

$$\xrightarrow{(H5)} \langle \mu , u_0 \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}$$

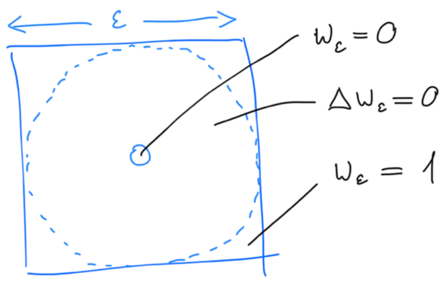
$\square$

Example:

$i \in \varepsilon \cdot \mathbb{Z}^N \subset \mathbb{R}^N$, $T^i_\varepsilon = \overline{B_{r_\varepsilon}(i)}$ closed balls around i,

$r_\varepsilon \ll \varepsilon$. Define w_ε by

$$\begin{cases} w_\varepsilon = 0 & \text{in } T^i_\varepsilon \\ \Delta w_\varepsilon = 0 & \text{in } B_\varepsilon(i) \setminus T^i_\varepsilon \\ w_\varepsilon = 1 & \text{in corners} \\ w_\varepsilon & \text{continuous} \end{cases}$$



In polar coordinates:

$$w_\varepsilon(r) = \frac{r_\varepsilon^{2-N} - r^{2-N}}{r_\varepsilon^{2-N} - \varepsilon^{2-N}} \qquad (N \geq 3)$$

Define μ by

$$\mu = \frac{|\partial B_1(0)| \, (N-2)}{2^N} \qquad (\mu \text{ is just a number!})$$