7 Nonlinear Wave Equations: Classical Existence and Uniqueness

From our previous discussions, we have built some considerable intuition behind how solutions of linear wave equations behave. In addition, we have studied numerous viewpoints and tools for analysing waves. From here on, we now turn our attention to nonlinear wave equations, i.e., partial differential equations of the form

$$\Box\phi = \mathcal{N}(t, x, \phi, \partial\phi),$$

where $\partial\phi$ denotes the *spacetime* gradient of ϕ, and where $\mathcal{N}(t, x, \phi, \partial\phi)$ is some nonlinear function of ϕ and $\partial\phi$. As mentioned before, our understanding of linear waves will be useful, as we work in perturbative settings in which nonlinear waves behave like linear ones.

Of course, there are countless possibilities for the nonlinearity $\mathcal{N}$, even within the subset of equations relevant to physics. However, in order to achieve some mathematical understanding of nonlinear waves, it is convenient to restrict our attention to model nonlinearities, in particular those which "scale consistently". Of particular interest to mathematicians are the following classes of nonlinear wave equations:

1. *NLW*: $\mathcal{N}(t, x, \phi, \partial\phi)$ is some power of ϕ, that is,[32]

$$\Box\phi = \pm|\phi|^{p-1}\phi, \qquad p > 1. \tag{7.1}$$

2. *dNLW*: $\mathcal{N}(t, x, \phi, \partial\phi)$ is some power of $\partial\phi$,

$$\Box\phi = (\partial\phi)^p, \qquad p > 1, \tag{7.2}$$

where $(\partial\phi)^p$ refers to some quantity that involves p-th powers of $\partial\phi$.

In order to keep our discussions concrete, we restrict our attention in this chapter to derivative nonlinear waves (dNLW), in the specific case $p = 2$. In particular, we consider the case when $\mathcal{N}(\phi, \partial\phi)$ is a quadratic form in $\partial\phi$:[33]

$$\mathcal{N}(t, x, \phi, \partial\phi) := Q(\partial\phi, \partial\phi).$$

In the remaining chapters, we restrict our stuides to the following initial value problem,

$$\Box\phi = Q(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \phi_0 : \mathbb{R}^n \to \mathbb{R}, \qquad \partial_t\phi|_{t=0} = \phi_1 : \mathbb{R}^n \to \mathbb{R}. \tag{7.3}$$

In this chapter, we show that (7.3) has unique local-in-time solutions. In the subsequent chapter, we explore the existence of global solutions, in particular for small initial data.

7.1 The ODE Perspective

From here on, it will be useful to view (7.3) as an analogue of the ODE setting—that ϕ is a curve, parametrised by the time t, taking values in some infinite-dimensional space X of real-valued functions on $\mathbb{R}^n$. Similar to ODEs, this is captured by considering ϕ as an element of the space $C(I; X)$ of continuous X-valued functions, where I is some interval

[32] For various reasons related to the qualitative behaviours of solutions, the "+" case in (7.1) is called *defocusing*, while the "−" case is called *focusing*.

[33] To avoid technical issues, we avoid settings in which $\mathcal{N}$ fails to be smooth.

containing the initial time $t_0 := 0$. In the ODE setting, X is simply the finite-dimensional space $\mathbb{R}^d$, with d the number of unknowns. On the other hand, in the PDE setting, one has far more freedom (and pitfalls) in the choice of the space X.

The upshot of this ambiguity is that one must choose X carefully. In particular, since $\phi(t)$ must lie in X for each time t, then X must be chosen such that its properties are propagated by the evolution of (7.3). Furthermore, we wish to apply the contraction mapping theorem (Theorem 1.4), hence it follows that X must necessarily be complete.

Recall that for linear wave equations, one has energy-type estimates in the Sobolev spaces $H^s(\mathbb{R}^n)$; see (5.26). Since one expects nonlinear waves to approximate linear waves in our setting, then one can reasonably hope that taking $X = H^s(\mathbb{R}^n)$ is sufficient to solve (7.3), at least for some values of s. Recall also that it was sometimes useful to consider both ϕ and $\partial_t\phi$ as unknowns in an equivalent first-order system,

$$\phi \in C(I; X), \qquad \partial_t\phi \in C(I; X').$$

From (5.26), we see it is reasonable to guess $X := H^{s+1}$ and $X' = H^s$. Returning to the viewpoint of a single second-order equation, the above can then be consolidated as

$$\phi \in C^0(I; H^{s+1}) \cap C^1(I; H^s).$$

The main objective of this chapter is to show that the above intuition can be validated, at least over a sufficiently small time interval. Over small times, one expects that the nonlinearity does not yet have a chance to significantly affect the dynamics, hence this seems reasonable. For large times, the nonlinearity could potentially play a significant role, in which case the above reasoning would collapse.

7.1.1 Strong and Classical Solutions

Before stating the main result, one must first discuss in more detail what is meant by a "solution" of (7.3). Recall that in the ODE theory, one works not with the differential equation (1.1) itself, but rather with an equivalent integral equation (1.3). For analogous reasons, one wishes to do the same in our current setting. In particular, this converts (7.3) into a fixed point problem, which one then solves by generating a contraction mapping.

The direct analogue of (1.3) would be the following first-order system:

$$\partial_t \begin{bmatrix} \phi(t) \\ \partial_t\phi(t) \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} + \int_0^t \begin{bmatrix} \partial_t\phi(s) \\ -\Delta_x\phi(s) - Q(\partial\phi, \partial\phi)(s) \end{bmatrix} ds. \tag{7.4}$$

However, this description immediate runs into problems, most notably with the desire to propagate the H^{s+1}-property for all $\phi(t)$. Indeed, suppose the pairs $(\phi(s), \partial_t\phi(s))$, where $0 \leq s < t$, are presumed to lie in $H^{s+1} \times H^s$. Then, the $\Delta_x\phi(s)$'s live in H^{s-1}, hence (7.4) implies that $(\phi(t), \partial_t\phi(t))$ only lie in the (strictly larger) space $H^s \times H^{s-1}$. Consequently, (7.4) is incompatible with the H^s-propagation that we desire.

The resolution to this issue is to use a more opportunistic integral equation that is more compatible with the H^s-propagation, namely, the representation which yielded the energy estimates (5.26) in the first place. For our specific setting, the idea is to use Duhamel's formula, (5.25), but with F replaced by our current nonlinearity:[34]

$$\phi(t) = \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1 - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}[Q(\partial\phi, \partial\phi)](s)ds. \tag{7.5}$$

[34] In comparison to the ODE setting, this is the analogue of (1.17).

Since the operators $\cos(t\sqrt{\Delta})$ and $\sin(t\sqrt{-\Delta})$ do preserve H^s-regularity, (7.5) seems to address the shortcoming inherent in (7.4). The remaining issue is the nonlinearity $Q(\partial\phi, \partial\phi)$ and whether this multiplication also "preserves" H^s-regularity in a similar fashion. This is the main new technical content of this section (which we unfortunately will not have time to cover in detail, as it involves a fair bit of harmonic analysis).

As had been mentioned before, the PDE setting differs from the ODE setting in that our integral description (7.5) is no longer equivalent to (7.3). In particular (ignoring for now the contribution from the nonlinearity), (7.5) makes sense for functions ϕ which are not (classically) differentiable enough for (7.3) to make sense. As a result of this, we make the following definition generalising the notion of solution.

Definition 7.1. *Let $s \geq 0$. We say that $\phi \in C^0(I; H^{s+1}) \cap C^1(I; H^s)$ is a* strong solution *(in $H^{s+1} \times H^s$) of* (7.3) *iff ϕ satisfies* (7.5) *for all $t \in I$.*

One must of course demonstrate that this definition is sensible. First, one can easily show that any $H^{s+1} \times H^s$-strong solution is also a $H^{s'+1} \times H^{s'}$-strong solution for any $0 \leq s' \leq s$, hence the notion of strong solutions is compatible among all H^s-spaces. Furthermore, from distribution theory, one can also show that any strong solution of (7.3) which is C^2 is also a solution of (7.3) in the classical sense. In this way, strong solutions are a direct generalisation of classical solutions. For brevity, we omit the details of these derivations.

7.2 Local Existence and Uniqueness

With the preceding discussion in mind, we are now prepared to give a precise statement of our local existence and uniqueness theorem for (7.3):

Theorem 7.2 (Local existence and uniqueness). *Let $s > n/2$, and suppose*

$$\phi_0 \in H^{s+1}(\mathbb{R}^n), \qquad \phi_1 \in H^s(\mathbb{R}^n). \tag{7.6}$$

Then, there exists $T > 0$, depending only on n and $\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s}$, such that the initial value problem (7.3) *has a unique strong solution*

$$\phi \in C^0([-T, T]; H^{s+1}(\mathbb{R}^n)) \cap C^1([-T, T]; H^s(\mathbb{R}^n)). \tag{7.7}$$

In the remainder of this subsection, we prove Theorem 7.2.[35] Note throughout that the basic argument mirrors that of ODEs in Section 1. However, the technical steps are further complicated here, since many estimates that were previously trivial in the ODE setting now rely on various properties and estimates for H^s-spaces.

7.2.1 Proof of Theorem 7.2: Analytical Tools

Before engaging in the proof, let us first discuss some of the main tools used within:

- Existence is again achieved by treating it as a fixed point problem for an integral equation (though one uses the Duhamel representation rather than the direct integral equation). This fixed point is then found via the contraction mapping theorem.[36]

[35] Much of this discussion will be heavily based on the contents of [Selb2001, Ch. 5].
[36] Alternately, this can be done via Picard iteration.

- As before, the contraction mapping theorem only achieves conditional uniqueness—i.e., uniqueness within a closed ball of the space of interest. As such, one needs an additional, though similar, argument to obtain the full uniqueness statement.

The main contrast with the ODE setting is the estimates one obtains for the sizes of the solution and the initial data. In the ODE setting, this involves measuring the lengths of finite-dimensional vectors, a relatively simple task. However, in the PDE setting, this involves measuring H^s-norms. As such, an additional layer of technicalities is required in order to understand and apply the toolbox of available estimates for these norms.

These H^s-estimates can be divided into two basic categories. The first are linear estimates, referring to estimates that are used to treat linear wave equations. For the current setting, this refers specifically to the energy estimate (5.26) from our previous discussions, which will be used to treat the main, non-perturbative part of the solution.

The remaining category contains nonlinear estimates, which, in the context of Theorem 7.2, refers to H^s-estimates for the nonlinearity $Q(\partial\phi, \partial\phi)$. Technically speaking, this is the main novel component of the proof that has not been encountered before. To be more specific, we will use the following classical estimate:

Theorem 7.3 (Product estimate). *If $f, g \in H^\sigma$, where $\sigma > n/2$, then $fg \in H^\sigma$, and*

$$\|fg\|_{H^\sigma} \lesssim \|f\|_{H^\sigma}\|g\|_{H^\sigma}. \tag{7.8}$$

Theorem 7.3 is in fact a direct consequence of the following estimates:

Theorem 7.4. *The following estimates hold:*

1. Product estimate*: If $\sigma \geq 0$ and $f, g \in L^\infty \cap H^\sigma$, then $fg \in H^\sigma$, and*

$$\|fg\|_{H^\sigma} \lesssim \|f\|_{L^\infty}\|g\|_{H^\sigma} + \|f\|_{H^\sigma}\|g\|_{L^\infty}. \tag{7.9}$$

2. Sobolev embedding*: If $f \in H^\sigma$ and $\sigma > n/2$, then $f \in L^\infty$, and*

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^\sigma}. \tag{7.10}$$

While Sobolev embedding is a much more general topic, the special case (7.10) has a simple and concise proof. The main step is to write f in terms of its Fourier transform:

$$|f(x)| \simeq \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)d\xi \right| \lesssim \left[\int_{\mathbb{R}^n} (1+|\xi|^2)^{-\sigma} d\xi \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} (1+|\xi|^2)^{\sigma} |\hat{f}(\xi)|^2 d\xi \right]^{\frac{1}{2}}.$$

The first integral on the right-hand side is finite since $\sigma > n/2$, while the second integral is precisely the H^σ-norm. Since the above holds for all $x \in \mathbb{R}^n$, then (7.10) is proved.

The product estimate (7.9) is considerably more involved, and unfortunatly only a brief and basic discussion of the ideas can be presented here. For detailed proofs, the reader is referred to [Selb2001, Ch. 5, 6] and [Tao2006, App. A].

First, one should note that the case $\sigma = 0$ is trivial by Hölder's inequality, and that the case $\sigma = 1$ then follows from the Leibniz rule:

$$\|\nabla_x(fg)\|_{L^2} \leq \|f \cdot \nabla_x g\|_{L^2} + \|\nabla_x f \cdot g\|_{L^2} = \|f\|_{L^\infty}\|\nabla_x g\|_{L^2} + \|\nabla_x f\|_{L^2}\|g\|_{L^\infty}.$$

For more general differential operators $\mathcal{D}$, in particular $\mathcal{D} := (1-\Delta)^{\sigma/2}$, one no longer has the Leibniz rule $\mathcal{D}(fg) = f \cdot \mathcal{D}g + \mathcal{D}f \cdot g$, hence the above simple argument no longer works. In spite of this, one can still recover something "close to the Leibniz rule", and this observation can be exploited to arrive at (7.9). However, capturing this property requires some creative use of Fourier transforms and harmonic analysis.

The rough idea is to understand how different frequences of f and g interact with each other in the product. For this, one applies what is called a *Littlewood-Paley decomposition*, in which a function h is decomposed into "frequency bands", i.e., components $P_k h$ whose Fourier transform is supported on the annulus $|\xi| \simeq 2^k$.[37] In particular, this decomposition is applied to both f and g, as well as the product fg. Then, each component $\mathcal{D}P_m(P_k f \cdot P_l g)$ can be treated separately, depending on the relative sizes of k, l, m. If done carefully enough, one can sum the estimates for each such component and recover (7.9).

Remark 7.5. Product estimates such as (7.9) form the beginning of a more general area of study known as *paradifferential calculus.*

7.2.2 Proof of Theorem 7.2: Existence

For convenience, we adopt the abbreviation

$$\mathcal{X} := C^0([-T,T]; H^{s+1}) \cap C^1([-T,T]; H^s),$$

and we denote the corresponding norm for $\mathcal{X}$ by

$$\|u\|_{\mathcal{X}} = \sup_{0 \leq t \leq T} [\|u(t)\|_{H^{s+1}} + \|\partial_t u(t)\|_{H^s}].$$

Furthermore, let $A > 0$ (to be chosen later), and consider the closed ball

$$Y = \{\phi \in \mathcal{X} \mid \|\phi\|_{\mathcal{X}} \leq A\}.$$

Since $\mathcal{X}$ is a Banach space, Y (with the metric induced by the $\mathcal{X}$-norm) forms a closed metric space. Consider now the map $\Psi : \mathcal{X} \to \mathcal{X}$ given by

$$[\Psi(\phi)](t) := \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1 - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}[Q(\partial\phi, \partial\phi)](s)ds. \quad (7.11)$$

Then, finding a strong solution of (7.3) is equivalent to finding a fixed point of Ψ.

Similar to the ODE setting, we accomplish this by showing that for large enough A and small enough T, depending on n and the size of the initial data, we have:

1. Ψ maps Y into itself.

2. $\Psi : Y \to Y$ is a contraction.

It suffices to prove (1) and (2), since then the contraction mapping theorem furnishes a fixed point for $\Psi \in Y$, which would complete the proof for existence in Theorem 7.2.

For (1), suppose $\phi \in Y$. By definition, $\Psi(\phi)$ satisfies the wave equation

$$\Box\Psi(\phi) = Q(\partial\phi, \partial\phi), \qquad \Psi(\phi)|_{t=0} = \phi_0, \qquad \partial_t\Psi(\phi)|_{t=0} = \phi_1,$$

[37] Essentially, applying a derivative to $P_k h$ is like multiplying by a constant 2^k in Fourier space.

at least in the Duhamel formula sense of (7.11).[38] As a result, one can apply the linear estimate (5.26), with $F := Q(\partial\phi, \partial\phi)$, in order to obtain[39]

$$\begin{aligned}
&\|\nabla_x[\Psi(\phi)](t)\|_{H^s} + \|\partial_t[\Psi(\phi)](t)\|_{H^s} \\
&\quad \leq C\left[\|\nabla_x\phi_0\|_{H^s} + \|\phi_1\|_{H^s} + \left|\int_0^t \||\partial\phi(\tau)|^2\|_{H^s} d\tau\right|\right] \\
&\quad \leq C\left[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + CT \sup_{\tau\in[-T,T]} \||\partial\phi(\tau)|^2\|_{H^s}\right],
\end{aligned} \tag{7.12}$$

for each $t \in [-T, T]$. Here, we use C to denote a positive constant that can change from line to line. Applying (7.8) to the nonlinear term in (7.12), we see for each t that

$$\begin{aligned}
&\|\nabla_x\Psi(\phi)(t)\|_{H^s} + \|\partial_t\Psi(\phi)(t)\|_{H^s} \\
&\quad \leq C\left[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + T \sup_{\tau\in[-T,T]} \|\partial\phi(\tau)\|_{H^s}^2\right] \\
&\quad \leq C[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + TA^2],
\end{aligned} \tag{7.13}$$

where in the last step, we recalled that $\phi \in Y$. Furthermore, from the fundamental theorem of calculus, we can control $\Psi(\phi)$ by $\partial_t\Psi(\phi)$: we have for each t that

$$\begin{aligned}
\|\Psi(\phi)(t)\|_{L^2} &\leq \|\Phi(\phi)(0)\|_{L^2} + T \sup_{\tau\in[-T,T]} \|\partial_t\Psi(\phi)(\tau)\|_{L^2} \\
&\leq C(1+T)[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + TA^2].
\end{aligned} \tag{7.14}$$

Since the size $M := \|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s}$ is fixed, then by choosing A to be much larger than M and T to be sufficiently small, we see from (7.13) and (7.14) that

$$\|\Psi(\phi)(t)\|_{H^{s+1}} + \|\partial_t(\Psi)(t)\|_{H^s} \leq A, \qquad t \in [-T, T]. \tag{7.15}$$

Taking a supremum in t, it follows that $\|\Psi(\phi)\|_{\mathcal{X}} \leq A$, and hence Ψ indeed maps any $\phi \in Y$ into Y. This completes the proof of property (1).

The proof of property (2) uses the same set of tools. Supposing $\phi, \tilde{\phi} \in Y$, then the difference $\Psi(\phi) - \Psi(\tilde{\phi})$ satisfies, again in the Duhamel sense, the wave equation

$$\Box[\Psi(\phi) - \Psi(\tilde{\phi})] = Q(\partial\phi, \partial\phi) - Q(\partial\tilde{\phi}, \partial\tilde{\phi}) = Q(\partial(\phi - \tilde{\phi}), \partial\phi) + Q(\partial\tilde{\phi}, \partial(\phi - \tilde{\phi})),$$

with zero initial data. Thus, applying (5.26) and then (7.8) to this quantity yields

$$\begin{aligned}
&\|\nabla_x\Psi(\phi) - \Psi(\tilde{\phi})](t)\|_{H^s} + \|\partial_t\Psi(\phi) - \Psi(\tilde{\phi})](t)\|_{H^s} \\
&\quad \leq CT \sup_{\tau\in[-T,T]} [\||\partial\phi||\partial(\phi - \tilde{\phi})|(\tau)\|_{H^s} + \||\partial\tilde{\phi}||\partial(\phi - \tilde{\phi})|(\tau)\|_{H^s}] \\
&\quad \leq CT \sup_{\tau\in[-T,T]} [\|\partial\phi(\tau)\|_{H^s} + \|\partial\tilde{\phi}(\tau)\|_{H^s}]\|\partial(\phi - \tilde{\phi})(\tau)\|_{H^s} \\
&\quad \leq CTA\|\phi - \tilde{\phi}\|_{\mathcal{X}}.
\end{aligned} \tag{7.16}$$

Another application of the fundamental theorem of calculus then yields

$$\|[\Psi(\phi) - \Psi(\tilde{\phi})](t)\|_{L^2} \leq T \sup_{\tau\in[-T,T]} \|\partial_t[\Psi(\phi) - \Psi(\tilde{\phi})](\tau)\|_{L^2} \leq CT^2A\|\phi - \tilde{\phi}\|_{\mathcal{X}}. \tag{7.17}$$

[38] The above also holds in the classical sense when ϕ is sufficiently smooth.

[39] The main point is that the proof of (5.26) uses only the Duhamel representation of waves.

Combining (7.16) and (7.17) and taking a supremum over $t \in [-T, T]$, we see that

$$\|\Psi(\phi) - \Psi(\tilde{\phi})\|_{\mathcal{X}} \leq CT(1+T)A\|\phi - \tilde{\phi}\|_{\mathcal{X}}.$$

Taking T to be sufficiently small yields

$$\|\Psi(\phi) - \Psi(\tilde{\phi})\|_{\mathcal{X}} \leq \frac{1}{2}\|\phi - \tilde{\phi}\|_{\mathcal{X}},$$

and hence $\Psi : Y \to Y$ is indeed a contraction. This concludes the proof of (2).

7.2.3 Proof of Theorem 7.2: Uniqueness

Suppose $\phi, \tilde{\phi} \in \mathcal{X}$ are solutions of (7.3). Then, $\psi := \phi - \tilde{\phi}$ solves, in the Duhamel sense,

$$\Box\psi = Q(\partial(\phi - \tilde{\phi}), \partial\phi) + Q(\partial\tilde{\phi}, \partial(\phi - \tilde{\phi})), \qquad \psi|_{t=0} = \partial_t\psi|_{t=0} = 0.$$

For convenience, we consider only $t \geq 0$; the $t < 0$ case is proved in an analogous manner. Similar to the preceding proof of existence, we apply (5.26) and (7.8) to obtain

$$\begin{aligned} &\|\nabla_x\psi(t)\|_{H^s} + \|\partial_t\psi(t)\|_{H^s} \\ &\quad\lesssim \int_0^t [\||\partial\phi||\partial(\phi - \tilde{\phi})|(\tau)\|_{H^s} + \||\partial\tilde{\phi}||\partial(\phi - \tilde{\phi})|(\tau)\|_{H^s}]d\tau \\ &\quad\lesssim \int_0^t [\|\partial\phi(\tau)\|_{H^s} + \|\partial\tilde{\phi}(\tau)\|_{H^s}]\|\partial\psi(\tau)\|_{H^s} d\tau. \end{aligned} \tag{7.18}$$

Since $[-T, T]$ is compact, by continuity and the definition if $\mathcal{X}$, the quantities

$$\|\partial\phi(t)\|_{H^s} + \|\partial\tilde{\phi}(t)\|_{H^s}, \qquad t \in [-T, T]$$

are uniformly bounded, and it follows that

$$\|\nabla_x\psi(t)\|_{H^s} + \|\partial_t\psi(t)\|_{H^s} \lesssim \int_0^t [\|\nabla_x\psi(\tau)\|_{H^s} + \|\partial_t\psi(\tau)\|_{H^s}]d\tau. \tag{7.19}$$

An application of the Gronwall inequality (1.12) yields that

$$\|\nabla_x\psi(t)\|_{H^s} + \|\partial_t\psi(t)\|_{H^s} = 0, \qquad t \in [0, T]. \tag{7.20}$$

Another application of the fundamental theorem of calculus then yields

$$\|\psi(t)\|_{L^2} = 0, \qquad t \in [0, T], \tag{7.21}$$

completing the proof of uniqueness.

7.3 Additional Comments

Finally, we address some additional issues related to Theorem 7.2.

7.3.1 Unconditional Uniqueness

Note that unconditional uniqueness in Theorem 7.2 is only a minor issue, as it is addressed using essentially the same tools as the proof of existence. However, there do exist other settings in which unconditional uniqueness becomes nontrivial, or even an open problem. [40]

[40] For instance, low-regularity existence results that use Strichartz-type estimates would obtain uniqueness only in a "Strichartz subspace" of $\mathcal{X}$. Establishing uniqueness in all of $\mathcal{X}$ requires new arguments.

7.3.2 Maximal Solutions

Another similarity with the ODE setting is that the time of existence in Theorem 7.2 again depends only on the *size* of the initial data. One consequence of this is that the proofs of Corollaries 1.11 and 1.12 can be directly carried over to the current setting. Thus, we have:

Corollary 7.6 (Maximal solutions). *Assume the hypotheses of Theorem 7.2. Then, there is a "maximal" interval* (T_-, T_+)*, where* $-\infty \leq T_- < 0 < T_+ \leq \infty$*, such that:*

- *There exists a strong solution* $\phi \in C^0((T_-, T_+); H^{s+1}) \cap C^1((T_-, T_+); H^s)$ *to* (7.3).
- ϕ *is the only strong solution to* (7.3) *on the interval* (T_-, T_+).
- *If* $\tilde{\phi} \in C^0(I; H^{s+1}) \cap C^1(I; H^s)$ *is another solution of* (7.3)*, then* $I \subseteq (T_-, T_+)$.

Furthermore, if $T_+ < \infty$*, then*

$$\limsup_{t \nearrow T_+} [\|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi\|_{H^s}] = \infty. \tag{7.22}$$

An analogous statement holds for T_-.

One can also view (7.22) as a continuation criterion for (7.3): if the left-hand side of (7.22) is instead finite, then one can extend the solution further in time beyond T_+. As a result of this, it is useful to establish results that replace (7.22) by some other quantity that can be more easily checked. One such classical result is the following:

Corollary 7.7 (Breakdown criterion). *Assume the hypotheses of Theorem 7.2, and let* $\phi : (T_-, T_+) \times \mathbb{R}^n \rightarrow \mathbb{R}$ *be the maximal solution to* (7.3)*. If* $T_+ < \infty$*, then*

$$\|(\partial_t \phi, \nabla_x \phi)\|_{L^\infty([0,T_+) \times \mathbb{R}^n)} = \infty. \tag{7.23}$$

Furthermore, an analogous statement holds for T_-.

Proof. Suppose instead that

$$\|(\partial_t \phi, \nabla_x \phi)\|_{L^\infty([0,T_+) \times \mathbb{R}^n)} < \infty. \tag{7.24}$$

Similar to the proof of Theorem 7.2, we apply (5.26) to (7.3), with $F := Q(\partial\phi, \partial\phi)$, along with the fundamental theorem of calculus for $\|\phi(t)\|_{L^2}$ to obtain the bound

$$\|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi(t)\|_{H^s} \lesssim (1 + T) \left[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + \int_0^t \| |\partial\phi(\tau)|^2 \|_{H^s} d\tau \right]$$

for each $0 \leq t < T_+$. Applying (7.9) and then (7.24) yields

$$\begin{aligned} &\|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi(t)\|_{H^s} \\ &\quad \lesssim (1 + T) \left[\|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + \int_0^t \|\partial\phi(\tau)\|_{L^\infty} \|\partial\phi(\tau)\|_{H^s} d\tau \right] \\ &\quad \lesssim (1 + T) \left\{ \|\phi_0\|_{H^{s+1}} + \|\phi_1\|_{H^s} + \int_0^t [\|\phi(\tau)\|_{H^{s+1}} + \|\partial_t \phi(\tau)\|_{H^s}] d\tau \right\} . \end{aligned}$$

By the Gronwall inequality (1.12), one uniformly bounds $\|\phi(t)\|_{H^{s+1}} + \|\partial_t \phi(t)\|_{H^s}$ for every $t \in [0, T_+)$. Corollary 7.6 then implies that $T_+ = \infty$. □

7.3.3 Finite Speed of Propagation

We revisit the finite speed of propagation property for homogeneous linear wave equations, Corollary 5.19. This property extends to many nonlinear waves, including (7.3).

Corollary 7.8 (Local uniqueness). *Let* $\phi, \tilde{\phi} \in C^2([-T,T] \times \mathbb{R}^n)$ *solve*

$$\Box\phi = Q(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \phi_0, \qquad \partial_t\phi|_{t=0} = \phi_1,$$
$$\Box\tilde{\phi} = Q(\partial\tilde{\phi}, \partial\tilde{\phi}), \qquad \tilde{\phi}|_{t=0} = \tilde{\phi}_0, \qquad \partial_t\tilde{\phi}|_{t=0} = \tilde{\phi}_1.$$

Moreover, fix $x_0 \in \mathbb{R}^n$ *and* $0 < R \leq T$, *and suppose* ϕ_i *and* $\tilde{\phi}_i$, *where* $i = 0, 1$, *are identical on* $\overline{B_{x_0}(R)}$. *Then,* ϕ *and* $\tilde{\phi}$ *are identical in the region*

$$\mathcal{C} = \{(t,x) \in [-R,R] \times \mathbb{R}^n \mid |x - x_0| \leq R - |t|\}.$$

Remark 7.9. In fact, both the local energy theory and Corollary 7.8 hold in the H^s-setting of this chapter. For brevity, we avoid proving this here.

Proof. Let $\psi := \phi - \tilde{\phi}$, which solves

$$\Box\psi = F := Q(\partial\psi, \partial\phi) + Q(\partial\tilde{\phi}, \partial\psi), \qquad \psi|_{t=0} = \psi_0, \qquad \partial_t\psi|_{t=0} = \psi_1,$$

with ψ_0 and ψ_1 vanishing on $\overline{B_{x_0}(R)}$. We then see from (5.29), with the above F, that

$$\begin{aligned} \mathcal{E}_{\psi,x_0,R}(t) &\leq \mathcal{E}_{\psi,x_0,R}(0) + C\int_0^t \int_{B_{x_0}(R-\tau)} (|\partial\phi(\tau)| + |\partial\tilde{\phi}(\tau)|)|\partial\psi(\tau)|^2 d\tau \\ &\leq C[\|\partial\phi\|_{L^\infty(\mathcal{C})} + \|\partial\tilde{\phi}\|_{L^\infty(\mathcal{C})}] \int_0^t \mathcal{E}_{\psi,x_0,R}(\tau) d\tau. \end{aligned} \tag{7.25}$$

for any $0 \leq t < R$, where $\mathcal{E}_{\psi,x_0,R}(t)$ is as defined in (5.28).

By compactness, the L^∞-norms in (7.25) are finite, hence (1.12) implies $\mathcal{E}_{\psi,x_0,R}(t) = 0$ for all $0 \leq t < R$. An analogous result can also be shown to hold for negative times $-R < t \leq 0$. By the fundamental theorem of calculus, ψ vanishes on all of $\mathcal{C}$. □

Remark 7.10. If $R > T$ in Corollary 7.8, the conclusion still holds in the truncated cone

$$\mathcal{C}_T = \{(t,x) \in [-T,T] \times \mathbb{R}^n \mid |x - x_0| \leq R - |t|\}.$$

The proof is a slight modification of the above.

7.3.4 Lower Regularity

For Theorem 7.2 and all the discussions above, we required that the initial data for (7.3) lie in $H^{s+1} \times H^s$ for $s > n/2$. One can then ask is whether there must still exist local solutions for less regular initial data, with $s \leq n/2$. Note that to find such solutions, one would require new ingredients in the proof, since one can no longer rely on Theorem 7.3.

Thus, any results that push s down to and below $n/2$ would require some mechanism to make up for the lack of spatial derivatives. The key observation is the time integral on the right-hand side of (7.5), which one can interpret as an antiderivative with respect to t. Note the wave equation $\partial_t^2 u = \Delta u$ can be interpreted as begin able to trade space derivatives for time derivatives, and vice versa. As a result, one can roughly think of being able to convert this antiderivative in t into antiderivatives in x. The effect is manifested in a class of estimates for wave equation known as *Strichartz estimates.*

Using such Strichartz estimates, one can reduce the regularity required in Theorem 7.2 to $s > n/2 - 1/2$ whenever $n \geq 3$.[41] For expositions on Strichartz estimates for wave equations, see [Selb2001,Tao2006]. To further reduce the needed regularity in (7.3), one requires instead *bilinear estimates* for the wave equation. In many cases, if Q in (7.3) has particularly favourable structure,[42] then one can further push down the required regularity. Lastly, for sufficiently small s, local existence of solutions to (7.3) is false; see, e.g., [Selb2001, Ch. 9].

[41] When $n = 2$, Strichartz estimates reduce the required regularity to $s > n/2 - 1/4$.

[42] In particular, if Q is a null form; see the upcoming chapter.