5 Linear Wave Equations

One of the main topics of this course is the study of *wave equations*. Throughout, we use the term "wave equations" to describe a broad class of PDEs, both linear and nonlinear, whose principal part consists of the wave operator,

$$\Box := -\partial_t^2 + \Delta_x := -\partial_t^2 + \sum_{k=1}^{n} \partial_{x^k}^2 . \tag{5.1}$$

Historically, wave equations arose in the study of vibrating strings. Since then, wave equations have played fundamental roles in both mathematics and physics. For instance:

- Wave equations serve as prototypical examples of a wider class of PDEs known as *hyperbolic PDEs*. Solutions of hyperbolic PDEs share a number of fundamental properties, such as finite speed of propagation. Thus, understanding the basic wave equation is an important step in studying hyperbolic PDEs in greater generality.
- Wave (and, more generally, hyperbolic) equations can be found in many fundamental equations of physics, including the Maxwell equations of electromagnetism and the Einstein field equations of general relativity. Therefore, in order to better grasp these physical theories, one must understand phenomena arising from wave equations.

The study of nonlinear wave equations in general is very difficult, with many important questions still left unanswered. However, research efforts over the past few decades have proved vastly fruitful in situations where the nonlinear theory as a perturbation of the linear theory. These refer to settings in which solutions of the nonlinear wave equation behave very closely to solutions of the linearised equation. In particular, the nonlinear theory we will eventually consider in these notes will be perturbative in this sense.

Thus, before tackling nonlinear wave equations, one must first understand the theory of *linear wave equations*. This is the main topic of this chapter. More specifically, we will discuss the *initial value*, or *Cauchy*, problem for both of the following:

1. *Homogeneous* linear wave equation: we look for a solution $\phi : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}$ of
$$\Box\phi = 0, \qquad \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1, \tag{5.2}$$
where $\phi_0, \phi_1 : \mathbb{R}^n \to \mathbb{R}$ comprise the *initial data*.
2. *Inhomogeneous* linear wave equation: we look for a solution $\phi : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}$ of
$$\Box\phi = F, \qquad \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1, \tag{5.3}$$
where ϕ_0, ϕ_1 are as before, and $F : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}$ is the forcing term.

Even the linear equations (5.2) and (5.3) have a rich theory. There are multiple viewpoints that one may adopt when studying these equations, which, broadly speaking, one can separate into "physical space" and "Fourier space" methods. As we shall see, different methods will prove useful in extracting different properties of solutions. In this chapter, we focus on this diversity of methods, and we compare and contrast the types of information that can be gleaned from each of these methods.

For convenience, we will adopt the following notations throughout this chapter:

- We write $A \lesssim_{c_1,\dots,c_m} B$ to mean $A \leq CB$ for some constant depending on $c_1, \dots, c_m$. When no constants c_k are given, the constant C is presumed to be universal.

- Unless otherwise specified, we assume various spaces of functions are over $\mathbb{R}^n$—for instance, $L^p := L^p(\mathbb{R}^n)$ and $H^s := H^s(\mathbb{R}^n)$. Similar conventions hold for the corresponding norms: $\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\mathbb{R}^n)}$ and $\| \cdot \|_{H^s} := \| \cdot \|_{H^s(\mathbb{R}^n)}$.
- We let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of smooth, rapidly decreasing functions:

$$\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n) \mid |\nabla_x^k f(x)| \leq (1 + |x|)^{-N} \text{ for all } k, N \geq 0 \}.$$

- Given $x_0 \in \mathbb{R}^n$ and $R > 0$, we define the ball and sphere about x_0 of radius R in $\mathbb{R}^n$:

$$B_{x_0}(R) := \{ x \in \mathbb{R}^n \mid |x - x_0| < R \}, \qquad S_{x_0}(R) := \{ x \in \mathbb{R}^n \mid |x - x_0| = R \}.$$

  In particular, we view the unit sphere $\mathbb{S}^{n-1}$ as the embedded sphere $S_0(1) \subseteq \mathbb{R}^n$.

To be precise, unless otherwise specified, integrals over $S_{x_0}(R)$ will be with respect to the volume forms induced from $\mathbb{R}^n$. Sometimes, this volume form is denoted by $d\sigma$.

5.1 Physical Space Formulas

The first method we discuss is to derive explicit formulas for the solutions of the homogeneous wave equation (5.2) in physical space. By "physical space" formulas, we mean formulas for the solution ϕ itself, as functions of the Cartesian coordinates t and x (as opposed to formulas for the Fourier transform, spatial or spacetime, of ϕ).

Explicit formulas for ϕ are nice in that they provide very direct information. For instance, these immediately imply the all-important finite speed of propagation properties of waves (as well as the strong Huygens principle for odd n). Moreover, from a more careful examination of these equations, one can also derive various asymptotic decay properties of waves.

On the other hand, what is not at all apparent from these formulas are the L^2-based properties of waves, i.e., conservation of energy and other energy-type estimates. These are not only fundamental properties, but they are also robust, in that they carry over to the analysis of nonlinear waves as well as waves on other backgrounds besides $\mathbb{R} \times \mathbb{R}^n$. In particular, methods which are heavily reliant on explicit formulas for (5.2) will likely fail to be useful in other more general settings of interest. As a result of this, we give only an abridged treatment of this method for completeness; the reader is referred to Section 2.4 of [Evan2002] for more detailed developments.

Another unfortunate feature of these explicit formulas is that they differ largely depending on the dimension n. As a result, we will have to treat different dimensions separately.

5.1.1 $n = 1$: D'Alembert's Formula

In this case, one can solve (5.2) using a simple change of variables. More specifically, rather than t and x, we consider instead the *null coordinates*,

$$u = t - x, \qquad v = t + x.$$

In these coordinates, the wave equation can expanded as

$$\partial_u \partial_v \phi = -\Box \phi = 0, \tag{5.4}$$

One can then integrate (5.4) directly. This yields

$$\partial_v \phi = G(v), \qquad \phi = \int_{-u}^{v} G(s) ds + H(u), \qquad \partial_u \phi = G(-u) + H'(u), \tag{5.5}$$

for some functions G and H (we integrate G from $-u$ here for later convenience). Both G and H in (5.4) can then be determined by the initial data (note that $x = -u = v$ when $t = 0$, and that $\partial_t = \partial_u + \partial_v$); a brief computation then yields

$$G(s) = \frac{1}{2}\phi_1(s) + \frac{1}{2}\phi_0'(s), \qquad H(s) = \phi_0(-s). \tag{5.6}$$

Combining (5.5) and (5.6), we arrive at *d'Alembert's formula*:

Theorem 5.1 (D'Alembert's formula). *Consider the problem* (5.2)*, with* $n = 1$*. If* $\phi_0 \in C^2(\mathbb{R})$ *and* $\phi_1 \in C^1(\mathbb{R})$*, then the function* $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, defined*

$$\phi(t,x) = \frac{1}{2}[\phi_0(x-t) + \phi_0(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(\xi) d\xi, \tag{5.7}$$

is C^2 *and solves* (5.2)*.*

Proof. By a direct computation, one can verify that ϕ in (5.7) indeed solves (5.2). □

Remark 5.2. Note the discussion preceding the statement of Theorem 5.1 serves as a proof of uniqueness for solutions of (5.2). Indeed, any solution of (5.2) that is regular enough so that the above manipulations are well-defined must in fact be the function (5.7). The same observation holds for the formulas in higher dimensions discussed below.

5.1.2 Odd $n > 1$: Method of Spherical Means

Unfortunately, in higher dimensions, one cannot concoct a similar change of variables to obtain an equation that can be integrated directly. However, there is a trick which reduces the problem, at least for odd spatial dimensions, to that of the previous case $n = 1$.

Roughly, the main idea is to write the wave equation in spherical coordinates,

$$-\partial_t^2\phi + \partial_r^2\phi + \frac{n-1}{r}\partial_r\phi + r^{-2}\not{\Delta}\phi = 0, \tag{5.8}$$

with $\not{\Delta}$ denoting the Laplace operator on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$. Now, if we integrate (5.8) over a sphere about the origin at a fixed time (i.e., over a level set of (t,r)), then the divergence theorem eliminates the spherical Laplacian. The resulting equation for these spherical averages of ϕ is then very close to the $(1+1)$-dimensional wave equation, for which we can solve using d'Alembert's formula. Moreover, this process can be repeated for spheres centered around any point $x \in \mathbb{R}^n$, not just the origin.

Because the main step of the process is this averaging of ϕ over spheres, this trick is usually referred to as the *method of spherical means*. Here, we briefly summarise the process in the case $n = 3$, for which the formulas remain relatively simple. We only state without derivation the result for higher (odd) dimensions.

For any point $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ and $r \in \mathbb{R}$, we define

$$M(t,x,r) := \frac{1}{4\pi}\int_{\mathbb{S}^2} \phi(t, x+ry) d\sigma(y),$$

where $d\sigma$ denotes the surface measure of the unit sphere $\mathbb{S}^2$. In particular, $M(t,x,r)$ represents the mean of ϕ over a sphere about (t,x) of radius $|r|$. Furthermore, one computes (noting that the spherical integral kills the spherical Laplacian) that

$$-\partial_t^2(rM) + \partial_r^2(rM) = 0,$$

Thus, rM satisfies the (1+1)-dimensional wave equation in t and r (for any fixed $x \in \mathbb{R}^3$), hence rM can be expressed explicitly using d'Alembert's formula. Also, by definition,

$$\phi(t,x) := \lim_{r \to 0} M(t,x,r).$$

Combining all the above, we arrive at *Kirchhoff's formula*:

Theorem 5.3 (Kirchhoff's formula). *Consider the problem* (5.2)*, with* $n = 3$*. Assuming* $\phi_0 \in C^3(\mathbb{R}^3)$ *and* $\phi_1 \in C^2(\mathbb{R}^3)$*, then the function* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$*, defined*

$$\begin{aligned}\phi(t,x) &= \frac{1}{4\pi} \int_{\mathbb{S}^2} [\phi_0(x+ty) + ty \cdot \nabla_x \phi_0(x+ty) + t \cdot \phi_1(x+ty)] d\sigma(y) \\ &= \frac{1}{4\pi t^2} \int_{S_x(|t|)} [\phi_0(y) + (y-x) \cdot \nabla_y \phi_0(y) + t \cdot \phi_1(y)] d\sigma(y).\end{aligned} \tag{5.9}$$

is C^2 *and solves* (5.2)*.*

Again, this can be proved by directly verifying that (5.9) satisfies (5.2).

For odd $n > 3$, an explicit solution can be derived by a similar process via spherical means. For brevity, we merely state the result here:

Theorem 5.4. *Consider the problem* (5.2)*, with* $n > 1$ *being odd. If* $\phi_0 \in C^{(n+3)/2}(\mathbb{R}^3)$ *and* $\phi_1 \in C^{(n+1)/2}(\mathbb{R}^3)$*, then the function* $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$*, defined*

$$\phi(t,x) = \frac{1}{\gamma_n} \partial_t (t^{-1}\partial_t)^{\frac{n-3}{2}} \left[t^{-1} \int_{S_x(|t|)} \phi_0 \right] + \frac{1}{\gamma_n} (t^{-1}\partial_t)^{\frac{n-3}{2}} \left[t^{-1} \int_{S_x(|t|)} \phi_1 \right], \tag{5.10}$$

is C^2 *and solves* (5.2)*, where* $\gamma_n := [1 \cdot 3 \cdot \dots \cdot (n-2)] \cdot |\mathbb{S}^{n-1}|$*.*

5.1.3 Even n: Method of Descent

For even dimensions, the main idea is to convert this to an odd-dimensional problem by adding a "dummy" variable $x_{n+1} \in \mathbb{R}$, with both ϕ_0 and ϕ_1 independent of this x_{n+1}. Thinking of this as an $(n+1)$-dimensional problem, we can now apply the previous results of Theorems 5.3 and 5.4. This is known as the *method of descent*.

Again, we summarise this process only for $n = 2$, for which the formulas are relatively simple. We now add a dummy variable $x^3 \in \mathbb{R}$, and we define

$$\tilde{\phi}_0(t,x,x^3) := \phi_0(t,x), \qquad \tilde{\phi}_1(t,x,x^3) := \phi_1(t,x).$$

Letting $x' = (x, x^3)$, then by (5.9), we see that the solution to (5.2), in the case $n = 3$ with initial data $\tilde{\phi}_0$ and $\tilde{\phi}_1$, is given by

$$\tilde{\phi}(t,x') = \frac{1}{4\pi} \int_{\mathbb{S}^2} [\tilde{\phi}_0(x'+ty') + ty' \cdot \nabla_{x'} \tilde{\phi}_0(x'+ty') + t \cdot \tilde{\phi}_1(x'+ty')] d\sigma(y'). \tag{5.11}$$

Now, the integrals over the hemispheres $x^3 > 0$ and $x^3 < 0$ of $\mathbb{S}^2$ can be written as weighted integrals of the unit disk in $\mathbb{R}^2$. Indeed, one can compute that for any $f : \mathbb{S}^2 \to \mathbb{R}$,

$$\int_{\mathbb{S}^2 \cap \{x^3 > 0\}} f(y') d\sigma(y') = \int_{B_0(1)} \frac{f(y^1, y^2, \sqrt{1-|y|^2})}{\sqrt{1-|y|^2}} dy, \tag{5.12}$$

$$\int_{\mathbb{S}^2 \cap \{x^3 < 0\}} f(y') d\sigma(y') = \int_{B_0(1)} \frac{f(y^1, y^2, -\sqrt{1-|y|^2})}{\sqrt{1-|y|^2}} dy. \tag{5.13}$$

Thus, combining (5.11) and (5.12), and recalling the definitions of $\tilde{\phi}_0$ and $\tilde{\phi}_1$, we obtain

$$\tilde{\phi}(t, x') = \frac{1}{2\pi} \int_{B_0(1)} \frac{\phi_0(x+ty) + ty \cdot \nabla_x \phi_0(x+ty) + t \cdot \phi_1(x+ty)}{\sqrt{1-|y|^2}} dy. \tag{5.14}$$

Note that $\tilde{\phi}$ is in fact independent of x^3. Thus, defining

$$\phi(t, x) := \tilde{\phi}(t, x'),$$

it follows that ϕ solves (5.2), for $n = 2$ and for initial data ϕ_0 and ϕ_1:

Theorem 5.5 (Method of descent). *Consider the problem* (5.2)*, in the case $n = 2$. If $\phi_0 \in C^3(\mathbb{R}^2)$ and $\phi_1 \in C^2(\mathbb{R}^2)$, then the function $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, defined*

$$\begin{aligned} \phi(t, x) &= \frac{1}{2\pi} \int_{B_0(1)} \frac{\phi_0(x+ty) + ty \cdot \nabla_x \phi_0(x+ty) + t \cdot \phi_1(x+ty)}{\sqrt{1-|y|^2}} dy & (5.15) \\ &= \frac{1}{2\pi t} \int_{B_x(|t|)} \frac{\phi_0(y) + (y-x) \cdot \nabla_y \phi_0(y) + t \cdot \phi_1(y)}{\sqrt{t^2 - |y-x|^2}} dy, & (5.16) \end{aligned}$$

is C^2 and solves (5.2)*.*

Again, one can verify that (5.15) is a solution of (5.2) through direct computation. Finally, we state without proof the formulas for the solution of (5.2) in all even dimensions:

Theorem 5.6. *Consider the problem* (5.2)*, with n being even. If $\phi_0 \in C^{(n+4)/2}(\mathbb{R}^n)$ and $\phi_1 \in C^{(n+2)/2}(\mathbb{R}^n)$, then the function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, defined*

$$\begin{aligned} \phi(t, x) &= \frac{1}{\gamma_n} \partial_t (t^{-1}\partial_t)^{\frac{n-2}{2}} \left[\int_{B_x(|t|)} \frac{\phi_0(y)}{\sqrt{t^2 - |y-x|^2}} dy \right] \\ &\quad + \frac{1}{\gamma_n} (t^{-1}\partial_t)^{\frac{n-2}{2}} \left[\int_{B_x(|t|)} \frac{\phi_1(y)}{\sqrt{t^2 - |y-x|^2}} dy \right], \end{aligned} \tag{5.17}$$

is C^2 and solves (5.2)*, and where $\gamma_n := (2 \cdot 4 \cdot \dots \cdot n) \cdot |B_0(1)|$.*

5.1.4 Finite Speed of Propagation

A fundamental property of waves that can be immediately seen from the physical space formulas (5.7), (5.10), and (5.17) is *finite speed of propagation.* If one alters the initial data ϕ_0, ϕ_1 in a small region, then that change travels in the solution ϕ at a finite speed, so that ϕ will not change at any point "far away" from where ϕ_0 and ϕ_1 were changed.

To be more illustrative, suppose we first consider trivial initial data, $\phi_0 = \phi_1 \equiv 0$. Then, the solution ϕ to (5.2) is simply the zero function, $\phi(t, x) = 0$. Next, suppose we alter ϕ_0 and ϕ_1, so that they are now nonzero on the unit ball $|x| < 1$. Then, applying the appropriate physical space formula, we see that at time $t = 1$, the solution $\phi(1, x)$ can be nonzero only when $|x| < 2$; this is because $\phi(1, x)$ is expressed as an integral of ϕ_0 and ϕ_1 on a ball or sphere of radius 1 about x. More generally, $\phi(t, x)$, at a time t, can be nonzero only when $|x| < 1 + |t|$. Thus, any change to ϕ_0 and ϕ_1 propagates at most at finite speed 1.

We give more precise statements of this below:

Theorem 5.7 (Finite speed of propagation). *Suppose ϕ is a solution of* (5.2)*, with ϕ_0 and ϕ_1 satisfying the assumptions of Theorem 5.1, 5.4, or 5.6, depending on n. In addition, we fix a point $x_0 \in \mathbb{R}^n$ and a radius $R > 0$.*

- *If the supports of ϕ_0 and ϕ_1 are contained in the ball $\overline{B_{x_0}(R)}$, then for any $t \in \mathbb{R}$, the support of $\phi(t, \cdot)$ is contained in the ball $\overline{B_{x_0}(R + |t|)}$.*
- *If ϕ_0 and ϕ_1 vanish on $\overline{B_{x_0}(R)}$, then ϕ vanishes on the cone*

$$C = \{(t, x) \in (-R, R) \times \mathbb{R}^n \mid |x - x_0| \leq R - |t|\}.$$

Proof. These can be directly observed using (5.7), (5.10), and (5.17). □

If n is odd, then we have an even stronger property. Indeed, from (5.9) and (5.10), we see that at time $t > 0$, the formula for $\phi(t, x)$ is expressed as an integral of ϕ_0 and ϕ_1 on a *sphere* of radius t about x. In other words, a change in ϕ_0 and ϕ_1 at a point $x_0 \in \mathbb{R}^n$ propagates entirely along the cone $\mathcal{C} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid |t| = |x - x_0|\}$. This is known in physics literature as the *strong Huygens principle*.

Theorem 5.8 (Strong Huygens principle). *Suppose ϕ solves* (5.2)*, with n odd. Then, if the first $(n-1)/2$ derivatives of ϕ_0 and the first $(n-3)/2$ derivatives of ϕ_1 vanish on the sphere $S_{x_0}(|t|)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then $\phi(t, x_0) = 0$.*

5.2 Fourier Space Formulas

We now discuss representation formulas for the solutions of (5.2) in Fourier space, that is, for the spatial Fourier transform of ϕ. In contrast to the physical space formulas in the previous subsection, the Fourier space formulas have the same format in all dimensions. This is an advantage in that one can use these formulas in the same way independently of dimension. On the other hand, this also means the Fourier representation hides many of the important qualititative differences among the formulas (5.7), (5.9), (5.10), (5.15), (5.17).

Suppose now that ϕ_0 and ϕ_1 are "nice enough" functions such that their Fourier transforms $\hat{\phi}_i : \mathbb{R}^n \to \mathbb{C}$ exist, and that after making "reasonable" transformations to $\hat{\phi}_0$ and $\hat{\phi}_1$, their inverse Fourier transforms also still exist.[23] Suppose ϕ solves (5.2), and let $\hat{\phi}$ denote its spatial Fourier transform, i.e., the Fourier transform of ϕ of only the x-variables:

$$\hat{\phi} : \mathbb{R}_t \times \mathbb{R}^n_\xi \to \mathbb{C}, \qquad \hat{\phi}(t, \xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(t, x) dx.$$

In particular, taking the spatial Fourier transform of (5.2) yields

$$-\partial_t^2 \hat{\phi}(t, \xi) - |\xi|^2 \hat{\phi}(t, \xi) \equiv 0, \qquad \hat{\phi}|_{t=0} = \hat{\phi}_0, \qquad \partial_t \hat{\phi}|_{t=0} = \hat{\phi}_1.$$

For each $\xi \in \mathbb{R}^n$, the above is a second-order ODE in t, which can be solved explicitly:

$$\hat{\phi}(t, \xi) = \cos(t|\xi|)\hat{\phi}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{\phi}_1(\xi). \tag{5.18}$$

This is the general representation formula for ϕ in Fourier space. Taking an inverse Fourier transform of (5.18) (assuming it exists) yields a formula for the solution ϕ itself. In particular, this inverse Fourier transform exists when both ϕ_0 and ϕ_1 lie in L^2.

For concise notation, one usually denotes this formula for ϕ via the operators

$$f \mapsto \cos(t\sqrt{-\Delta})f = \mathcal{F}^{-1}[\cos(t|\xi|)\mathcal{F}f], \qquad f \mapsto \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f = \mathcal{F}^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}f\right],$$

corresponding to multiplication by $\cos(t|\xi|)$ and $|\xi|^{-1}\sin(t|\xi|)$ in Fourier space.[24] Thus, from (5.18) and the above considerations, we obtain:

[23] To sidestep various technical issues, we avoid the topic of distributional solutions.

[24] There are spectral-theoretic justifications for such notations, but we will not discuss these here.

Theorem 5.9. *Consider the problem* (5.2)*, for general* n*, and suppose* $\phi_0, \phi_1 \in L^2$*. Then, the solution* ϕ *to* (5.2) *can be expressed as*

$$\phi(t) = \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1. \tag{5.19}$$

Remark 5.10. Note that in general, the right-hand side of (5.19) may not be twice differentiable in the classical sense. Thus, one must address what is meant by (5.19) being a "solution". While there are multiple ways to characterise such "weak solutions", we note here that (5.19) does solve (5.2) in the sense of distributions.

As mentioned before, the physical and Fourier space formulas highlight rather different aspects of waves. For instance, the finite speed of propagation properties that were immediate from the physical space formulas cannot be readily seen from (5.19). On the other hand, it is easy to obtain L^2-type estimates for ϕ from (5.19) via Plancherel's theorem, while such estimates are not at all apparent from the physical space formulas:

Theorem 5.11. *Suppose* $\phi_0 \in H^{s+1}$ *and* $\phi_1 \in H^s$ *for some* $s \geq 0$*. Then, for any* $t \in \mathbb{R}$*, the solution* ϕ *to* (5.2) *satisfies the following estimate:*

$$\|\nabla_x \phi(t)\|_{H^s} + \|\partial_t \phi(t)\|_{H^s} \lesssim \|\nabla_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s}. \tag{5.20}$$

Proof. This follows from Plancherel's theorem and the definition of H^s-norms:[25]

$$\begin{aligned}
\|\nabla_x \phi(t)\|_{H^s} &\leq \|\cos(t|\xi|) \cdot (1+|\xi|^2)^{\frac{s}{2}} \cdot \xi\phi_0\|_{L^2} + \|\sin(t|\xi|) \cdot |\xi|^{-1}\xi \cdot (1+|\xi|^2)^{\frac{s}{2}} \cdot \phi_1\|_{H^s} \\
&\leq \|(1+|\xi|^2)^{\frac{s}{2}} \cdot \xi\phi_0\|_{L^2} + \|(1+|\xi|^2)^{\frac{s}{2}} \cdot \phi_1\|_{H^s} \\
&= \|\nabla_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s}.
\end{aligned}$$

Similarly, for $\partial_t \phi$, we have

$$\begin{aligned}
\|\partial_t \phi(t)\|_{H^s} &\leq \|\sin(t|\xi|) \cdot (1+|\xi|^2)^{\frac{s}{2}} \cdot |\xi|\phi_0\|_{L^2} + \|\cos(t|\xi|) \cdot (1+|\xi|^2)^{\frac{s}{2}} \cdot \phi_1\|_{H^s} \\
&\leq \|\nabla_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s}.
\end{aligned}$$

□

In particular, the H^s-regularity of (first derivatives of) solutions of (5.2) is propagated. In other words, as long as the hypotheses of Theorem 5.11 are satisfied, the curves $t \mapsto \nabla_x \phi(t)$ and $t \mapsto \partial_t \phi(t)$ lie in the (infinite-dimensional) space H^s.

Remark 5.12. From either the fundamental theorem of calculus or from Fourier space estimates, one can show that under the hypotheses of Theorem 5.11,

$$\|\phi(t)\|_{H^s} \lesssim 1 + |t|, \qquad t \in \mathbb{R}.$$

Moreover, this is nearly optimal, one can construct solutions ϕ of (5.2) such that $\|\phi(t)\|_{L^2}$ grows faster than $|t|^{1-\varepsilon}$ for any $\varepsilon > 0$; see Section 4.5 in [Selb2001].

Remark 5.13. Alternatively, one can rewrite the Fourier space formula (5.18) as

$$\hat{\phi}(t,\xi) = \frac{1}{2}e^{it|\xi|}[\hat{\phi}_0(\xi) - i|\xi|^{-1}\hat{\phi}_1(\xi)] + \frac{1}{2}e^{-it|\xi|}[\hat{\phi}_0(\xi) + i|\xi|^{-1}\hat{\phi}_1(\xi)].$$

This leads to the *half-wave decomposition* of ϕ:

$$\phi(t) = \frac{1}{2}e^{it\sqrt{-\Delta_x}}\phi_- + \frac{1}{2}e^{-it\sqrt{-\Delta_x}}\phi_+, \qquad \phi_\pm := \phi_0 \pm i(-\Delta_x)^{-\frac{1}{2}}\phi_1. \tag{5.21}$$

[25] Here, for generality and for convenience, we use the Fourier definition of H^s-norms.

In particular, this recasts (5.2) as solving the following first-order *half-wave equations*:

$$\partial_t \beta \pm \sqrt{-\Delta_x}\beta = 0, \qquad \beta|_{t=0} = \beta_0. \tag{5.22}$$

This half-wave representation is often better suited for harmonic analysis techniques.

5.3 Duhamel's Principle

We now turn our attention to the general inhomogeneous wave equation, (5.3). As was mentioned in Chapter 1, one could construct a solution of inhomogeneous linear ODE from a solution of the corresponding homogeneous ODE via Duhamel's principle, and this idea extends to PDEs as well. However, Duhamel's principle, as discussed, applied only to first-order equations, while the wave equation is of course second-order.

The trick for overcoming this issue to adopt $\psi := \partial_t \phi$ as a second unknown. By considering (ϕ, ψ) as the unknowns, then (5.3) can be written as a first-order system:

$$\partial_t \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ -\Delta_x \phi - F \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Delta_x & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} - \begin{bmatrix} 0 \\ F \end{bmatrix}, \qquad \begin{bmatrix} \phi \\ \psi \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix}. \tag{5.23}$$

Remark 5.14. Perhaps some clarification is needed for the term "first-order". In (5.23), we adopt the ODE-inspired perspective of treating the solution (ϕ, ψ) to the wave equation as a curve in an infinite-dimensional space of (pairs of) functions $\mathbb{R}^n \to \mathbb{R}$. In this viewpoint, the Laplacian Δ_x is thought of as a linear operator on this infinite-dimensional space.

Let L denote the *linear propagator* for the above system (5.23), that is,[26]

$$L(t) \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} := \begin{bmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{bmatrix}, \qquad t \in \mathbb{R}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ solve (5.23), *with* $F \equiv 0$. Then, from (5.19) and its t-dervative, we have

$$L(t) \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1 \\ \sin(t\sqrt{-\Delta})\sqrt{-\Delta}\phi_0 + \cos(t\sqrt{-\Delta})\phi_1 \end{bmatrix}. \tag{5.24}$$

Formally, one can repeat the derivation of Proposition 1.14, assuming the integrating factors process extends to our setting. This yields that the solution (ϕ, ψ) of (5.23) satisfies

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = L(t) \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} - \int_0^t L(t-s) \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds.$$

Finally, recalling the formula (5.24) for $L(t)$ and restricting our attention only to the first component (the solution ϕ of the original wave equation), we obtain:

Theorem 5.15 (Duhamel's principle (Fourier)). *Consider the problem* (5.3)*. Let* $\phi_0, \phi_1 \in L^2$*, and assume* $F \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^n_x))$*. Then, the solution* ϕ *can be written as*

$$\phi(t) = \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1 - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds. \tag{5.25}$$

Furthermore, if $\nabla_x \phi_0 \in H^s$*,* $\phi_1 \in H^s$*, and* $F \in L^\infty(\mathbb{R}; H^s)$*, then for any* $t \in \mathbb{R}$*,*

$$\|\nabla_x \phi(t)\|_{H^s} + \|\partial_t \phi(t)\|_{H^s} \lesssim \|\nabla_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s} + \left| \int_0^t \|F(\tau)\|_{H^s} d\tau \right|. \tag{5.26}$$

[26]For now, one simply assume that $\phi_0, \phi_1 \in \mathcal{S}$ for convenience.

Proof. Again, recalling the definitions of the operators in (5.25) in Fourier space, one can differentiate (5.25) and check directly that it satisfies (5.3). The estimate (5.26) follows from Plancherel's theorem in the same manner as (5.20). □

One can also derive analogues of Duhamel's principle using the physical space formulas. Consider, for concreteness, the case $n = 3$ (the other dimensions can be handled analogously). Let L_ϕ denote the projection of the linear propagator to its first (ϕ-)component. Then, assuming ϕ_0 and ϕ_1 to be sufficiently differentiable, $L_1(t)(\phi_0, \phi_1)$ is given by the right-hand side of (5.9). Duhamel's formula then yields the following:

Theorem 5.16 (Duhamel's principle (physical)). *Consider the problem* (5.3)*, with* $n = 3$*. If* $\phi_0 \in C^3(\mathbb{R}^3)$*,* $\phi_1 \in C^2(\mathbb{R}^3)$*, and* $F \in C^2(\mathbb{R} \times \mathbb{R}^3)$*, then the function*

$$\begin{aligned}\phi(t,x) &= \frac{1}{4\pi t^2} \int_{S_x(|t|)} [\phi_0(y) + (y-x) \cdot \nabla_y \phi_0(y) + t \cdot \phi_1(y)] d\sigma(y) \\ &\quad + \frac{1}{4\pi} \int_0^t \int_{S_x(|s|)} \frac{F(t-s,y)}{s} d\sigma(y) ds\end{aligned} \tag{5.27}$$

is C^2 *and satisfies* (5.3)*. Moreover, analogous formulas hold in other dimensions.*

5.4 The Energy Identity

By the "energy" of a wave ϕ, one generally refers to L^2-type norms for first derivatives of ϕ. Recall that (5.26) already provides such an energy estimate.[27] Below, we will show how physical space methods can be used to achieve more precise energy *identities*.

Furthermore, physical space methods will also allow us to *localise* energy identities and estimates within spacetime cones. This property is closely connected to finite speed of propagation and hence is far less apparent from Fourier space techniques.

To state the general local energy identity, we need a few more notations. Given $x_0 \in \mathbb{R}^n$:

- We denote by $\partial_{r(x_0)}$ the radial derivative centred about x_0,

$$\partial_{r(x_0)} := \frac{x - x_0}{|x - x_0|} \cdot \nabla_x .$$

- We denote by $\nabla\!\!\!/_{y(x_0)}$ the angular gradients on the spheres $S_{x_0}(R)$, $R > 0$.

With the above in mind, the local energy identity can now be stated as follows:

Theorem 5.17 (Local energy identity). *Let* ϕ *be a* C^2*-solution of* (5.3)*, and suppose* F *is continuous. Given* $x_0 \in \mathbb{R}^n$ *and* $R > 0$*, we define the* local energy *of* ϕ *by*

$$\mathcal{E}_{\phi,x_0,R}(t) := \frac{1}{2} \int_{B_{x_0}(R-t)} [|\partial_t \phi(t,x)|^2 + |\nabla_x \phi(t,x)|^2] dx, \qquad 0 \leq t < R, \tag{5.28}$$

Then, for any $0 \leq t_1 < t_2 < R$*, the following* local energy identity *holds:*

$$\mathcal{E}_{\phi,x_0,R}(t_2) + \mathcal{F}_{\phi,x_0,R}(t_1,t_2) = \mathcal{E}_{\phi,x_0,R}(t_1) - \int_{t_1}^{t_2} \int_{B_{x_0}(R-t)} F(t,x) \partial_t \phi(t,x) dx dt, \tag{5.29}$$

where $\mathcal{F}_{\phi,x_0,R}$ *is the corresponding* local energy flux*,*

$$\mathcal{F}_{\phi,x_0,R}(t_1,t_2) = \frac{1}{2} \int_{t_1}^{t_2} \int_{S_{x_0}(R-t)} [|(\partial_t - \partial_{r(x_0)})\phi(t,y)|^2 + |\nabla\!\!\!/_{y(x_0)} \phi(t,y)|^2] d\sigma(y) dt. \tag{5.30}$$

[27]Taking $s > 0$ in (5.26) yields higher-order energy estimates.

Proof. Writing the integral in $\mathcal{E}_{\phi,x_0,R}(t)$ in polar coordinates as

$$\mathcal{E}_{\phi,x_0,R}(t) := \frac{1}{2}\int_0^{R-t}\int_{S_{x_0}(r)}[|\partial_t\phi(t,y)|^2 + |\nabla_x\phi(t,y)|^2]d\sigma(y)dr,$$

we derive that

$$\begin{aligned}\mathcal{E}'_{\phi,x_0,R}(t) &= \int_{B_{x_0}(R-t)}[\partial_t^2\phi(t,x)\partial_t\phi(t,x) + \partial_t\nabla_x\phi(t,x)\cdot\nabla_x\phi(t,x)]dx \\ &\qquad - \frac{1}{2}\int_{S_{x_0}(R-t)}[|\partial_t\phi(t,y)|^2 + |\nabla_x\phi(t,y)|^2]d\sigma(y) \\ &= I_1 + I_2.\end{aligned}$$

The term I_1 can be further expanded using the wave equation (5.2):

$$\begin{aligned}I_1 &= \int_{B_{x_0}(R-t)}[-F(t,x)\partial_t\phi(t,x) + \Delta_x\phi(t,x)\partial_t\phi(t,x) + \partial_t\nabla_x\phi(t,x)\cdot\nabla_x\phi(t,x)]dx \\ &= I_{1,1} + I_{1,2} + I_{1,3}.\end{aligned}$$

Integrating $I_{1,2}$ by parts yields

$$I_{1,2} = -I_{1,3} + \int_{S_{x_0}(R-t)}\partial_{r(x_0)}\phi(t,x)\partial_t\phi(t,x).$$

Thus, combining the above results in the identity

$$\begin{aligned}\mathcal{E}'_{\phi,x_0,R}(t) &= -\frac{1}{2}\int_{S_{x_0}(R-t)}[|\partial_t\phi(t,y)|^2 + |\nabla_x\phi(t,y)|^2 - 2\partial_{r(x_0)}\phi(t,x)\partial_t\phi(t,x)]d\sigma(y) \\ &\qquad - \int_{B_{x_0}(R-t)}F(t,x)\partial_t\phi(t,x)dx.\end{aligned}$$

Recall now that the gradient of ϕ can be decomposed into its radial and angular parts, with respect to x_0. In terms of lengths, we have

$$|\nabla_x\phi|^2 = |\partial_{r(x_0)}\phi|^2 + |\not\nabla_{x(x_0)}\phi|^2.$$

As a result of some algebra, our identity for $\mathcal{E}'_{\phi,x_0,R}$ becomes

$$\begin{aligned}\mathcal{E}'_{\phi,x_0,R}(t) &= -\frac{1}{2}\int_{S_{x_0}(R-t)}[|\partial_t\phi(t,y) - \partial_{r(x_0)}\phi(t,y)|^2 + |\not\nabla_{x(x_0)}\phi(t,y)|^2]d\sigma(y) \\ &\qquad - \int_{B_{x_0}(R-t)}F(t,x)\partial_t\phi(t,x)dx.\end{aligned}$$

Integrating the above in t from t_1 to t_2 results in (5.29). $\square$

Remark 5.18. Since $\tilde{\phi}(t) := \phi(-t)$ also satsifies a wave equation, then under the assumptions of Theorem 5.17, an analogous result holds for negative times $-R < t \le 0$.

In the homogeneous case, one can use to Theorem 5.17 to almost immediately recover finite speed of propagation (though not the strong Huygens principle):

Corollary 5.19 (Finite speed of propagation). *Suppose ϕ is a C^2-solution of (5.2), and let $x_0 \in \mathbb{R}^n$ and $R > 0$. If ϕ_0 and ϕ_1 vanish on $\overline{B_{x_0}(R)}$, then ϕ vanishes on*

$$C = \{(t,x) \in (-R,R)\times\mathbb{R}^n \mid |x - x_0| \le R - |t|\}.$$

Proof. Noting that the flux (5.30) is always nonnegative, then (5.29) implies

$$\mathcal{E}_{\phi,x_0,R}(t) \leq \mathcal{E}_{\phi,x_0,R}(0) = 0, \qquad 0 \leq t < R,$$

where the initial energy vanishes due to our assumptions. Since these $\mathcal{E}_{\phi,x_0,R}(t)$ vanish, then both $\partial_t \phi$ and $\nabla_x \phi$ vanish on $C^+ := \{(t,x) \in C \mid t \geq 0\}$. Since $\phi(0)$ vanishes on $B_{x_0}(R)$, then the fundamental theorem of calculus implies ϕ also vanishes on C^+. A similar conclusion can be reached for negative times using time symmmetry. $\square$

Thus far, we have constructed, via both physical and Fourier space methods, solutions to (5.2) and (5.3). However, we have devoted only cursory attention to is the uniqueness of solutions to (5.2). Suppose ϕ and $\tilde{\phi}$ both solve (5.3) (with the same F, ϕ_0, ϕ_1). Then, $\phi - \tilde{\phi}$ solves (5.2), with zero initial data. Thus, applying Corollary 5.19 with various x_0 and R yields that $\phi = \tilde{\phi}$. As a result, we have shown:

Corollary 5.20 (Uniqueness). *If ϕ and $\tilde{\phi}$ are C^2-solutions to* (5.3)*, then $\phi = \tilde{\phi}$.*

Remark 5.21. In fact, Holmgren's theorem implies that solutions to (5.2) (and also to (5.3) for real-analytic F) are unique in the much larger class of distributions.[28]

5.4.1 Global Energy Identities

One can also use physical space methods to derive *global* energy bounds similar to (5.26), as long as there is sufficiently fast decay in spatial directions. This approach has an additional advantage in that one obtains an energy identity rather than just an estimate.

Theorem 5.22 (Energy identity). *Let ϕ be a C^2-solution of* (5.3)*, and suppose for any $t \in \mathbb{R}$ that $\nabla_x \phi(t)$, $\partial_t \phi(t)$, and $F(t)$ decay rapidly.*[29] *Define the* energy *of ϕ by*

$$\mathcal{E}_\phi(t) := \frac{1}{2} \int_{\mathbb{R}^n} [|\partial_t \phi(t,x)|^2 + |\nabla_x \phi(t,x)|^2] dx, \qquad t \in \mathbb{R}. \tag{5.31}$$

Then, for any $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, the following energy identity *holds:*

$$\mathcal{E}_\phi(t_2) = \mathcal{E}_\phi(t_1) - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} F \partial_t \phi \cdot dx dt \tag{5.32}$$

Proof. This follows by taking $R \nearrow \infty$ in (5.29) and by noticing that the local energy flux (5.30) vanishes in this limit due to our decay assumptions. $\square$

Remark 5.23. From (5.32), one can recover the energy estimate (5.26) with $s = 0$. Note that one also obtains higher-order energy identities from (5.32), since the wave operator commutes with $\langle -\Delta_x \rangle^s := (1 + |-\Delta_x|)^{\frac{s}{2}}$, and hence[30]

$$\square(\langle -\Delta_x \rangle^s \phi) = \langle -\Delta_x \rangle^s F.$$

Moreover, in the homogeneous case, one captures an even stronger statement:

[28]Holmgren's theorem is a classical result which states that for any linear PDE with analytic coefficients, solutions of the noncharacteristic Cauchy problem are unique in the class of distributions.

[29]More specifically, $|\partial_t \phi(t)| + |\nabla_x \phi(t)| + |F(t)| \lesssim (1 + |x|)^{-N}$ for any $N > 0$. Note that such assumptions for ϕ are not unreasonable, since Corollary 5.19 implies this holds for compactly supported initial data.

[30]Of course, one must assume additional differentiability for ϕ if $s > 0$.

Corollary 5.24 (Conservation of energy). *Let ϕ be a C^2-solution of* (5.2)*, and suppose for any $t \in \mathbb{R}$ that $\nabla_x \phi(t)$, $\partial_t \phi(t)$, and $F(t)$ decay rapidly. Then, for any $t \in \mathbb{R}$,*

$$\mathcal{E}_\phi(t) = \mathcal{E}_\phi(0) = \frac{1}{2} \int_{\mathbb{R}^n} [|\phi_1(x)|^2 + |\nabla_x \phi_0(x)|^2] dx. \tag{5.33}$$

5.4.2 Some Remarks on Regularity

Thus far, our energy identities have required that ϕ is at least C^2, which from our (physical space) representation formulas may necessitate even more regularity for ϕ_0 and ϕ_1. One can hence ask whether these results still apply when ϕ is less smooth.

In fact, one can recover this local energy theory (and, by extension, finite speed of propagation and uniqueness) for rough solutions ϕ of (5.2), arising from initial data $\phi_0 \in H^1_{\text{loc}}$ and $\phi_1 \in L^2_{\text{loc}}$. (For solutions of (5.3), one also requires some integrability assumptions for F.) In general, this is done by approximating ϕ_0 and ϕ_1 by smooth functions and applying the existing theory to the solutions arising from the regularised data. Then, by a limiting argument, one can transfer properties for the regularised solutions to ϕ itself.

The remainder of these notes will deal mostly with highly regular functions, for which all the methods we developed will apply. As a result, we avoid discussing these regularity issues here, as they can be rather technical and can obscure many of the main ideas.

5.5 Dispersion of Free Waves

While we have shown energy conservation for solutions ϕ of the homogeneous wave equation (5.2), we have not yet discussed how solutions decay in time. On one hand, the total energy of $\phi(t)$, given by the L^2-norms of $\partial_t \phi(t)$ and $\nabla_x \phi(t)$, does not change in t. However, what happens over large times is that the wave will propagate further outward (though at a finite speed), and the profile of $\phi(t)$ disperses over a larger area in space. Correspondingly, the magnitude of $|\phi(t)|$ will become smaller as the profile spreads out further.

A pertinent question is to ask what is the generic rate of decay of $|\phi(t)|$ as $|t| \nearrow \infty$. The main result, which is often referred to as a *dispersive estimate*, is the following:

Theorem 5.25 (Dispersive estimate). *Suppose ϕ solves* (5.2)*, with $\phi_0, \phi_1 \in \mathcal{S}$. Then,*

$$\|\phi(t)\|_{L^\infty} \leq C t^{-\frac{n-1}{2}}, \tag{5.34}$$

where the constant C depends on various properties of ϕ_0 and ϕ_1.

Remark 5.26. The representation formulas (5.7), (5.10), and (5.17) demonstrate that (5.34) is false if ϕ_0 and ϕ_1 do not decay sufficiently quickly.

One traditional method to establish Theorem 5.25 is by using harmonic analysis methods. Recalling the half-wave decomposition (5.21), Theorem 5.25 reduces to proving dispersion estimates for the half-wave propagators $e^{\pm it\sqrt{-\Delta_x}}$. These can be shown to be closely connected to decay properties for the Fourier transform of the surface measure of $\mathbb{S}^{n-1}$.

There also exist more recent physical space methods for deriving (5.34). Very roughly, these are based primarily on establishing weighted integral estimates for $\partial_t \phi$ and $\partial_x \phi$ over certain *spacetime* regions. While these methods require more regularity from ϕ_0 and ϕ_1, they have the additional advantage of being applicable to wave equations on backgrounds that are not $\mathbb{R} \times \mathbb{R}^n$; see the lecture notes of M. Dafermos and I. Rodnianski, [Dafe2013].