MA 18 FINAL PRACTICE EXAM

Collaboration or use of online help is *forbidden* on the whole practice test until Tuesday at noon. After that you can discuss it with your peers, TAs and instructors.

You can turn in the five problems marked with $\star$ for *extra credit.* You should turn them in to your TA in recitations on Tuesday, December 7 at noon. If you can't make it to the recitations put them into the section box in the Kassar House before Tuesday at noon. Other than the due time there is no time limit on how many hours you can spend on these problems. Write solutions nicely, explain your reasoning and make sure your handwriting is readable.

Completely correct solutions on these five problems can bring you extra 10 points on your final. The final will be worth 100 points.

(7) $\star$ Let W be the three-dimensional region under the graph of $f(x, y) = \exp(x^2 + y^2)$ and over the region in the plane defined by $1 \le x^2 + y^2 \le 2$.

(a) Find the volume of W.

(b) Find the flux of the vector field $\mathbf{F} = (2x - xy)\mathbf{i} - y\mathbf{j} + yz\mathbf{k}$ out of the boundary ∂W.

Solution.

(a)

$$vol(W) = \int_1^{\sqrt{2}} \int_0^{2\pi} e^{r^2} r \, d\theta \, dr = \pi \int_1^2 e^u \, du = \pi(e^2 - e)$$

(b)

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} \, dV = \iiint_W 1 \, dV = vol(W) = \pi(e^2 - e)$$

(8) Let $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$. Show that the integral of $\mathbf{F}$ around the circumference of the square $[0, 1] \times [0, 1]$ in the xy plane is 0 by

(a) direct evaluation.

(b) using Green's theorem.

Solution.

(a) Let us denote the four sides of the square $C_1, \ldots, C_4$ with C_1 being the side connecting $(0, 0)$ and $(0, 1)$ and the other sides denoted consecutively in a counterclockwise fashion. Parametrize C_1 using $\mathbf{r}(t) = (t, 0)$ with $0 \le t \le 1$. Then we get

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_0^1 (0, t^2) \cdot (1, 0) \, dt = 0.$$

Similarly we compute the work along the other three sides.

(b) Denote the perimeter C (with the appropriate orientation) and the square R. Using Green's Theorem, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA = \iint_R (2x - 2x) \, dA = 0.$$

(9) $\star$ Let

$$\mathbf{F}(x, y, z) = e^{xy} \left[(yz + xy^2 z)\mathbf{i} + (xz + x^2 yz)\mathbf{j} + xy\mathbf{k} \right].$$

(a) Show that $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^3 \to \mathbb{R}$ (there's no need to actually find f).

(b) Let C be the curve obtained by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x = 1/2$, and let S be the portion of the sphere with $x \ge 1/2$. Draw a figure including possible (compatible) orientations for C and S. State Stokes' theorem for this region.

(c) With $\mathbf{F}$ as in (a), let $\mathbf{G} = \mathbf{F} + (z - y)\mathbf{i} + y\mathbf{k}$. Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}$$

with your preferred orientation, where S is as in (b). Here, $\nabla\times\mathbf{G} = \operatorname{curl}\mathbf{G}$ and $d\mathbf{S} = \mathbf{n}\, dS$, where $\mathbf{n}$ is a unit normal to the surface. *Hint:* $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A}+\nabla\times\mathbf{B}$.

(d) Can you find f in (a)?

Solution.

(a) We first note that $\mathbf{F}$ is defined for all x, y, z, and that it is continuous, as are its partial derivatives. Therefore, if we verify that $\nabla\times\mathbf{F} = \mathbf{0}$, we can conclude that there indeed exists a potential function ϕ. We leave it for you to verify this...

(b) We omit the sketch in this solution. Stokes' Theorem always asserts that

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \int_{C+} \mathbf{F}\cdot d\mathbf{r}.$$

(c) Using the fact that $\nabla\times\mathbf{F} = \mathbf{0}$, we have that

$$\begin{aligned}\nabla\times\mathbf{G} &= \nabla\times(\mathbf{F}+(z-y)\mathbf{i}+y\mathbf{k})\\ &= \nabla\times\mathbf{F}+\nabla\times((z-y)\mathbf{i}+y\mathbf{k})\\ &= \mathbf{0}+\nabla\times((z-y)\mathbf{i}+y\mathbf{k}).\end{aligned}$$

Denote $\mathbf{H} = (z-y)\mathbf{i}+y\mathbf{k}$. We parametrize the perimeter C and invoke Stokes' Theorem. We first use the fact that $x = 1/2$ to discover that the curve C is given by $x = 1/2, y^2+z^2 = 3/4$. Therefore, a good parametrization is $\mathbf{r}(t) = (1/2, \sqrt{3}/2\ \cos t, \sqrt{3}/2\ \sin t)$, which implies that $\mathbf{r}'(t) = (0, -\sqrt{3}/2\ \sin t, \sqrt{3}/2\ \cos t)$. Therefore we finally have

$$\begin{aligned}\iint_S (\nabla\times\mathbf{G})\cdot\mathbf{n}\, dS &= \iint_S (\nabla\times\mathbf{H})\cdot\mathbf{n}\, dS = \int_{C+} \mathbf{H}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{H}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\, dt\\ &= \int_0^{2\pi} (\frac{\sqrt{3}}{2}\ \sin t - \frac{\sqrt{3}}{2}\ \cos t, 0, \frac{\sqrt{3}}{2}\ \cos t)\cdot(0, -\frac{\sqrt{3}}{2}\ \sin t, \frac{\sqrt{3}}{2}\ \cos t)\, dt\\ &= \int_0^{2\pi} \frac{\sqrt{3}}{2}\cos^2 t\, dt = \frac{\sqrt{3}\pi}{2}\end{aligned}$$

(d) (This is more difficult). By inspection we see that all components of $\mathbf{F}$ include the term e^{xy}. Therefore f must contain this term. Notice that $\frac{\partial}{\partial z}e^{xy} = 0$ whereas the third component of $\mathbf{F}$ is $F_3 = xy\, e^{xy}$. An expression of the form $f(x, y, z) = xyz\, e^{xy}$ might do the trick. A quick inspection shows that indeed $\nabla f = \mathbf{F}$. Note that f is not unique – we can add a constant to f and it would still be a potential function for $\mathbf{F}$.

(10) (a) Let S be the surface $x^2 + 2y^2 + 2z^2 = 1$. Find a parametrization of S and use it to find the tangent plane to S at $(1/\sqrt{2}, 1/2, 0)$.

(b) Verify that the curve $\mathbf{c}(t) = \cos t\ \mathbf{i} + (1/\sqrt{2})\sin t\ \mathbf{j}, 0 \le t \le 2\pi$ lies in the surface S, and that $\mathbf{c}'(\pi/4)$ lies in the tangent plane found in (a).

(c) Write down an integral representing the area of the surface using the parametrization you found.

Solution.

(a) When determining how to parametrize, we first identify the type of surface in question. In this case, we are given an ellipsoid. Probably, the parametrization will resemble that of a *sphere*, with the two parameters being $u = \phi$ and $v = \theta$. We just need to account for the fact that this ellipsoid is not perfectly symmetric. We see that it is altered along the y and z axes. We therefore insert constants in front of the usual y and z components of the parametrization to take this into account. We parametrize as follows:

$$\mathbf{r}(u, v) = (\sin u\cos v, \frac{\sqrt{2}}{2}\sin u\sin v, \frac{\sqrt{2}}{2}\cos u).$$

This indeed satisfies $x^2 + 2y^2 + 2z^2 = 1$.

Now, we calculate $\mathbf{r}_u, \mathbf{r}_v$ and their cross product, in order to find the normal direction.

$$\mathbf{r}_u = (\cos u \cos v, \frac{\sqrt{2}}{2}\cos u \sin v, -\frac{\sqrt{2}}{2}\sin u)$$

$$\mathbf{r}_v = (-\sin u \sin v, \frac{\sqrt{2}}{2}\sin u \cos v, 0)$$

$$\mathbf{r}_u \times \mathbf{r}_v = (\frac{1}{2}\sin^2 u \cos v, \frac{\sqrt{2}}{2}\sin^2 u \sin v, \frac{\sqrt{2}}{2}\cos u \sin u)$$

The point we are interested in is $(1/\sqrt{2}, 1/2, 0)$ which corresponds to $u = \pi/2$ and $v = \pi/4$. Plugging these u, v into the expression for $\mathbf{r}_u \times \mathbf{r}_v$ we get

$$\mathbf{r}_u \times \mathbf{r}_v = (\frac{\sqrt{2}}{4}, \frac{1}{2}, 0)$$

which is a normal vector to the plane in question. The last step is to plug into the plane equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

the values $(a, b, c) = (\frac{\sqrt{2}}{4}, \frac{1}{2}, 0)$ and $(x_0, y_0, z_0) = (1/\sqrt{2}, 1/2, 0)$ to get

$$\frac{\sqrt{2}}{4}x - \frac{1}{4} + \frac{1}{2}y - \frac{1}{4} = 0$$

or,

$$\sqrt{2}x + 2y = 2.$$

(b) The curve $\mathbf{c}(t) = \cos t\,\mathbf{i} + (1/\sqrt{2})\sin t\,\mathbf{j}, 0 \le t \le 2\pi$ indeed satisfies $x^2 + 2y^2 + 2z^2 = 1$ and therefore lies in the surface. Its tangent is given by

$$\mathbf{c}'(t) = (-\sin t, \frac{1}{\sqrt{2}}\cos t, 0)$$

and when plugging in $t = \pi/4$ we get $\mathbf{c}'(\pi/4) = (-\frac{\sqrt{2}}{2}, \frac{1}{2}, 0)$ which is perpendicular to the vector $\mathbf{r}_u \times \mathbf{r}_v$, and is therefore contained in the plane.

(c) Finally, we write an integral representing the area of the surface:

$$\iint_S 1\,dS = \int_0^{\pi}\int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\|\,dv\,du = \cdots$$

where we omit the calculation of the norm of $\mathbf{r}_u \times \mathbf{r}_v$ (note that you are not required to actually evaluate the integral, only to simplify as much as possible).

(11)⋆ Answer the following short questions: If true, justify, if false give a counterexample.

(a) The path integral $\int_{\mathbf{c}} 2\pi\,ds$ is the surface area of a cylinder of radius 1 and height 2π where the path is defined by $\mathbf{c} = (\cos t, \sin t, 0)$, and $0 \le t \le 2\pi$.

(b) There is no vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Solution.

(a) True. Denote the surface area in question S. Then

$$\int_{\mathbf{c}} 2\pi\,ds = 2\pi \times (\text{the arc length of } \mathbf{c}) = S.$$

(b) True. Recall that *the curl is divergence free.* Therefore, for any $\mathbf{F}$, it must hold that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. But $\nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3 \neq 0$.

(12) Let C be the circle $x^2 + y^2 = 1, z = 0$ and let

$$\mathbf{F}(x, y, z) = [x^2y^3 + y - \cos(x^3)]\mathbf{i} + [x^3y^2 + \sin(y^3) + x]\mathbf{j} + z\mathbf{k}$$

Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$.

Solution. Looks like a Stokes' Theorem type of question! We first check $\nabla \times \mathbf{F}$ to see if it is a more friendly expression. Indeed, we find that $\nabla \times \mathbf{F} = (0, 0, 0)$. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \iint_S 0\,dS = 0.$$