CARDIFF UNIVERSITY
PRIFYSGOL CAERDYDD

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Academic Year: 2023/24

Examination Period: Autumn

Module Code: MA3016

Examination Paper Title: Partial Differential Equations

Duration: 2 hours

Please read the following information carefully:

Structure of Examination Paper:

- There are 5 pages including this page.
- There are **4** questions in total.
- There are no appendices.
- The maximum mark for the examination paper is 100 and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Instructions for completing the examination:

- Complete the front cover of any answer books used.
- This examination paper must be submitted to an Invigilator at the end of the examination.
- Answer **THREE** questions.
- Each question should be answered on a separate page.

You will be provided with / or allowed:

- **ONE** answer book.
- The **use of calculators** is **not permitted** in this examination.
- The use of a translation dictionary between English or Welsh and another language, provided that it bears an appropriate departmental stamp, is permitted in this examination.
- The use of the student's own notes, up to **1 sheet (2 sides) of A4 paper**, is permitted in this examination.

1. **The Diffusion Equation.** Consider the *diffusion equation* on the real line:

$$u_t(x,t) - k u_{xx}(x,t) = 0, \quad -\infty < x < +\infty, \quad t > 0,$$

where $k > 0$ is fixed. Let $x_0 < x_1$ and $0 \leq t_0 < t_1$ and define the rectangle

$$R := [x_0, x_1] \times [t_0, t_1]$$

in the (x,t) plane. Define Γ to be the union of the bottom, right and left edges of R.

(a) State the maximum principle for R. [3]

(b) Prove the maximum principle for R. [20]
Solution: this was proved in class (notes of 27 Oct)

(c) For $a > 0$, solve the the diffusion equation with initial condition $\phi(x) = \begin{cases} 1 & |x| \leq a, \\ 0 & |x| > a. \end{cases}$

Express your answer in terms of the error function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp.$$

You may use the formula

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy.$$

[10]

Solution: this is Q2.4.1 from the book (which I solved)

2. **The Wave Equation.** Consider an infinite string with density $\rho > 0$ and tension $T > 0$ (both assumed to be constant). The associated wave equation is

$$u_{tt}(x,t) - \frac{T}{\rho}u_{xx}(x,t) = 0, \qquad -\infty < x < +\infty, \quad t > 0. \tag{*}$$

(a) What is the wave speed c? [3]
Solution: $c = \sqrt{T/\rho}$

(b) Let $x_0 \in \mathbb{R}$ and let $t_0 > 0$. Sketch and label clearly the domains of influence and of dependence of the point (x_0, t_0) in space-time, ensuring to specify the slopes of their boundaries. [5]
Solution: this was discussed in class (notes of 26 Oct)

(c) Let u be a solution of (*) and assume that u, u_t and u_x all tend to 0 as $x \to \pm\infty$. Prove that u does *not* satisfy a maximum principle. In your proof you may choose initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ so long as they satisfy that u, u_t and u_x all tend to 0 as $x \to \pm\infty$ (so, for instance, $\phi(x)$ *cannot* be periodic!) [10]
Solution: this was a homework problem (Q2.5.1), but see my solution because you are not allowed to take ϕ that is periodic (which is something many of you relied on in your solutions!)

(d) Consider the wave equation for a finite string with mixed Dirichlet/Neumann boundary conditions:

$$\begin{cases} u_{tt}(x,t) - \frac{T}{\rho}u_{xx}(x,t) = 0, & 0 < x < \ell, \quad t > 0, \\ u(0,t) = u_x(\ell,t) = 0, & t > 0. \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), & 0 < x < \ell. \end{cases}$$

i. Separate the variables $u(x,t) = X(x)T(t)$ to express u in series form (you may assume that the equation $-X'' = \lambda X$ with mixed Dirichlet/Neumann boundary conditions has only non-negative eigenvalues). [10]
Solution: this (is almost identical to) Q4.2.2 from the book (which I solved). The only difference is that the BCs are reversed

ii. If $\phi(x) = 5\sin(\frac{5\pi}{2\ell}x)$ and $\psi(x) = 0$, what are the coefficients in the preceding expansion? (You may use the fact that $\int_0^\ell \sin^2(\frac{5\pi}{2\ell}x)\,dx = \frac{\ell}{2}$ and that the eigenfunctions are mutually orthogonal without proof). [5]
Solution: this is very similar to the last part of Q5.1.9 from the book (which I solved)

3. **Properties of Differential Operators and First-Order PDEs.**

(a) Solve the first-order equation [8]

$$\begin{cases} u_x(x,y) + \cos x u_y(x,y) = 0, \\ u(0,y) = y^2. \end{cases}$$

Solution: this was solved in class (notes of 13 Oct)

(b) Let $\mathcal{L}$ be the operator given by $\mathcal{L}f(x) = f'(x)$ on the interval $(0,1)$ with Dirichlet boundary conditions (i.e. $f(0) = f(1)$). Find its eigenvalues and eigenfunctions, and show that the eigenfunctions are mutually orthogonal. [8]

Solution: this is Q5.3.6 from the book (which I solved)

(c) Let $\{X_n\}_{n=0}^{\infty}$ be the standard eigenfunctions of the operator $\mathcal{L}X = -X''$ on (a,b) with periodic boundary conditions. Let f be a real-valued, twice continuously differentiable function on $[a,b]$, with $f(a) = f(b)$ and $f'(a) = f'(b)$. Consider its Fourier series $f(x) = \sum_{n=1}^{\infty} A_n X_n(x)$. Prove that this series converges uniformly. [17]

Solution: this was proved in class (notes of 30 Nov)

4. **The Laplace Equation.**

(a) Let $a > 0$. Solve $\Delta u = 0$ in the disk $D = \{r < a\}$ with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$. [10]

Solution: this is Q6.3.2 from the book (which I solved)

(b) Let $D \subset \mathbb{R}^2$ be an open, bounded and connected set. Prove that any solution to the problem

$$\begin{cases} \Delta u = f & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$$

is unique. You may rely on the maximum principle in your proof. [10]

Solution: this was proved in class (notes of 7 Dec)

(c) Find the harmonic function $u(x, y)$ in the square

$$R = \{(x, y) \,|\, 0 < x < \pi, 0 < y < \pi\}$$

satisfying the boundary conditions $u_x(0, y) = u_x(\pi, y) = u(x, 0) = 0$ and $u(x, \pi) = g(x)$. You may assume that the equation $-X'' = \lambda X$ with Neumann boundary conditions has only non-negative eigenvalues. [13]

Solution: a very similar problem was solved in class, with slightly different BCs (notes of 8 Dec), also Q6.2.3 (which I solved) is very similar