CARDIFF UNIVERSITY
PRIFYSGOL CAERDYDD

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Academic Year: 2023

Examination Period: Autumn

Module Code: MA3016

Examination Paper Title: Partial Differential Equations

Duration: 2/3 hours

Please read the following information carefully:

Structure of Examination Paper:

- There are X pages including this page.
- There are **X** questions in total.
- The following appendices areappendix is attached to this examination paper:
  Statistical tables
  Some Fundamental Distributions and their Properties
- There are no appendices.
- The maximum mark for the examination paper is 100% and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Instructions for completing the examination:

- Complete the front cover of any answer books used.
- This examination paper must be submitted to an Invigilator at the end of the examination.
- Answer **THREE** questions.
- Each question should be answered on a separate page.

You will be provided with / or allowed:

- **ONE** answer book.
- Squared graph paper.
- The following items are provided as an Appendix: Statistical tables
- The **use of calculators** is **not permitted** in this examination.
- The use of a translation dictionary between English or Welsh and another language, provided that it bears an appropriate departmental stamp, is permitted in this examination.
- The use of the student's own notes, up to **1 sheet (2 sides) of A4 paper**, is permitted in this examination.

1. **The Wave Equation.** Consider an infinite string with density $\rho > 0$ and tension $T > 0$ (both assumed to be constant). The associated wave equation is

$$u_{tt}(x,t) - \frac{T}{\rho}u_{xx}(x,t) = 0, \qquad -\infty < x < +\infty, \quad t > 0. \tag{*}$$

(a) What is the wave speed c? [3]

(b) Assume that u, u_t and u_x all tend to 0 as $x \to \pm\infty$. Prove that the string's energy $E(t)$ is conserved, where [10]

$$E(t) := \frac{1}{2}\rho\int_{-\infty}^{\infty} u_t(x,t)^2\,dx + \frac{1}{2}T\int_{-\infty}^{\infty} u_x(x,t)^2\,dx.$$

(c) The *damped* wave equation for some damping constant $r > 0$ is

$$u_{tt}(x,t) - \frac{T}{\rho}u_{xx}(x,t) + ru_t(x,t) = 0, \qquad -\infty < x < +\infty, \quad t > 0.$$

Prove that in this case the energy may decrease over time. [10]

(d) Assume that a string satisfying the wave equation (*) is initially "plucked", i.e. with the initial conditions (for some fixed $a > 0$)

$$\begin{cases} u(x,0) = \phi(x) = \begin{cases} a - |x| & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a, \end{cases} \\ \\ u_t(x,0) = \psi(x) = 0, \qquad -\infty < x < +\infty. \end{cases}$$

i. When will the disturbance be felt at the point $b \in \mathbb{R}$, where $b > a$? [5]

ii. Will the string ever stop vibrating at the same point b? If so, when? Explain using d'Alembert's formula: [5]

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

2. **The Diffusion Equation.** Consider the *diffusion equation* in the interval $(0, \ell)$ with Dirichlet boundary conditions:

$$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0, & 0 < x < \ell, \quad t > 0, \\ u(0,t) = u(\ell,t) = 0, & t > 0, \\ u(x,0) = \phi(x), & 0 < x < \ell. \end{cases}$$

Assume that $k > 0$ and that the function ϕ is continuous on $[0, \ell]$, non-negative and not identically 0. Let $T > 0$ and define the rectangle

$$R := [0, \ell] \times [0, T]$$

in the (x,t) plane. Define Γ to be the union of the bottom, right and left edges of R.

(a) State the maximum principle for R. [3]

(b) State the *strong* maximum principle for R. [3]

(c) Use the energy method to prove that $\int_0^\ell u(x,t)^2\,dx$ is a *strictly* decreasing function of t. *Hint: multiply the equation by u and integrate.* [9]

(d) Separate the variables $u(x,t) = X(x)T(t)$ to express u in series form (you may assume that the equation $-X'' = \lambda X$ with Dirichlet boundary conditions has only positive eigenvalues). [9]

(e) If $\phi(x) = \sin(\frac{2\pi}{\ell}x)$, what are the coefficients in the preceding expansion? (You may use the fact that $\int_0^\ell \sin^2(\frac{2\pi}{\ell}x)\,dx = \frac{\ell}{2}$ and that the eigenfunctions are mutually orthogonal without proof). [9]

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3. **The Laplace Equation.**

(a) Let the function u be harmonic in a disk $B \subset \mathbb{R}^2$ of radius $a > 0$ centred at the origin, with $u = h(\theta)$ on ∂B. Poisson's formula is

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \, d\phi.$$

State and prove the *mean value property*. [5]

(b) Let $D \subset \mathbb{R}^2$ be an open, bounded and connected set. Let the function u be harmonic in D and continuous in $\overline{D} = D \cup \partial D$. State and prove the *strong maximum principle*. [14]

(c) Find the harmonic function $u(x, y)$ in the square

$$R = \{(x, y) \,|\, 0 < x < \pi, 0 < y < \pi\}$$

satisfying the boundary conditions $u(0, y) = u(\pi, y) = u(x, 0) = 0$ and $u(x, \pi) = g(x)$. You may assume that the equation $-X'' = \lambda X$ with Dirichlet boundary conditions has only positive eigenvalues. [14]

4. **Properties of Differential Operators and First-Order PDEs.**

(a) Solve the first-order equation [8]

$$\begin{cases} 5u_x(x,y) - 2u_y(x,y) = 0, \\ u(x,0) = \cos x. \end{cases}$$

(b) Let $\mathcal{L}$ be the operator given by $\mathcal{L}f(x) = -f''(x)$ on some interval (a,b) with either Dirichlet, Neumann or Periodic boundary conditions. Prove that $\mathcal{L}$ has only real eigenvalues, and that its eigenfunctions can be taken to be real-valued. In your proof you may use Green's second identity for two twice continuously differentiable functions $y_1(x), y_2(x)$ on (a,b), and continuous on $[a,b]$: [20]

$$\int_a^b (-y_1''\overline{y_2} + y_1\overline{y_2}'')\, dx = (-y_1'\overline{y_2} + y_1\overline{y_2}')\,|_{x=a}^{b}.$$

(c) If $\mathcal{L}$ is subject to Neumann boundary conditions, can 0 be an eigenvalue? Explain your answer. [5]