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Academic Year: 2023

Examination Period: Autumn

Module Code: MA3016

Examination Paper Title: Partial Differential Equations

Duration: 2/3 hours

Please read the following information carefully:

Structure of Examination Paper:

- There are X pages including this page.
- There are **X** questions in total.
- The following appendices areappendix is attached to this examination paper:
  Statistical tables
  Some Fundamental Distributions and their Properties
- There are no appendices.
- The maximum mark for the examination paper is 100% and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Instructions for completing the examination:

- Complete the front cover of any answer books used.
- This examination paper must be submitted to an Invigilator at the end of the examination.
- Answer **THREE** questions.
- Each question should be answered on a separate page.

You will be provided with / or allowed:

- **ONE** answer book.
- Squared graph paper.
- The following items are provided as an Appendix: Statistical tables
- The **use of calculators** is **not permitted** in this examination.
- The use of a translation dictionary between English or Welsh and another language, provided that it bears an appropriate departmental stamp, is permitted in this examination.
- The use of the student's own notes, up to **1 sheet (2 sides) of A4 paper**, is permitted in this examination.

1. **The Wave Equation.** Consider an infinite string with density $\rho > 0$ and tension $T > 0$ (both assumed to be constant). The associated wave equation is

$$u_{tt}(x,t) - \frac{T}{\rho} u_{xx}(x,t) = 0, \qquad -\infty < x < +\infty, \quad t > 0. \tag{*}$$

(a) What is the wave speed c? [3]

(b) Assume that u, u_t and u_x all tend to 0 as $x \to \pm\infty$. Prove that the string's energy $E(t)$ is conserved, where [10]

$$E(t) := \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t(x,t)^2 \, dx + \frac{1}{2}T \int_{-\infty}^{\infty} u_x(x,t)^2 \, dx.$$

(c) The *damped* wave equation for some damping constant $r > 0$ is

$$u_{tt}(x,t) - \frac{T}{\rho} u_{xx}(x,t) + r u_t(x,t) = 0, \qquad -\infty < x < +\infty, \quad t > 0.$$

Prove that in this case the energy may decrease over time. [10]

(d) Assume that a string satisfying the wave equation (*) is initially "plucked", i.e. with the initial conditions (for some fixed $a > 0$)

$$\begin{cases} u(x,0) = \phi(x) = \begin{cases} a - |x| & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a, \end{cases} \\ \\ u_t(x,0) = \psi(x) = 0, \qquad -\infty < x < +\infty. \end{cases}$$

i. When will the disturbance be felt at the point $b \in \mathbb{R}$, where $b > a$? [5]

ii. Will the string ever stop vibrating at the same point b? If so, when? Explain using d'Alembert's formula: [5]

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds.$$

a) The wave speed is $c = \sqrt{\frac{T}{\rho}}$.

b) To show that the energy is conserved we show that $E'(t) = 0$.

We compute:

$$E'(t) = \frac{d}{dt}\left(\frac{1}{2}\rho\int_{-\infty}^{\infty} u_t^2(x,t)\,dx\right) + \frac{d}{dt}\left(\frac{1}{2}T\int_{-\infty}^{\infty} u_x^2(x,t)\,dx\right)$$

$$= \frac{1}{2}\rho\int_{-\infty}^{\infty} 2u_t u_{tt}\,dx + \frac{1}{2}T\int_{-\infty}^{\infty} 2u_x u_{xt}\,dx$$

(Here we use the wave eq. itself: $\rho u_{tt} = T u_{xx}$)

$$= \underbrace{T\int_{-\infty}^{\infty} u_t u_{xx}\,dx}_{I} + T\int_{-\infty}^{\infty} u_x u_{xt}\,dx$$

We integrate the first term by parts:

$$I = \int_{-\infty}^{\infty} u_t \frac{\partial}{\partial x}(u_x)\,dx = -\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(u_t)\,u_x\,dx + \underbrace{[u_t u_x]_{x=-\infty}^{\infty}}$$

$$= -\int_{-\infty}^{\infty} u_{tx} u_x\,dx$$

This is 0 since these functions vanish at $\pm\infty$.

$$\Rightarrow \quad E'(t) = -T\int_{-\infty}^{\infty} u_{tx}u_x\,dx + T\int_{-\infty}^{\infty} u_x u_{xt}\,dx = 0.$$

Therefore the energy doesn't change over time $\rightarrow$ it is conserved.

c) We repeat the same computation in the case of the damped wave eq:

$$E'(t) = \rho\int_{-\infty}^{\infty} u_t u_{tt}\,dx + T\int_{-\infty}^{\infty} u_x u_{xt}\,dx$$

As before, we substitute u_{tt} using the wave equation, which now takes the form: $u_{tt} = \frac{T}{\rho}u_{xx} - r u_t$.

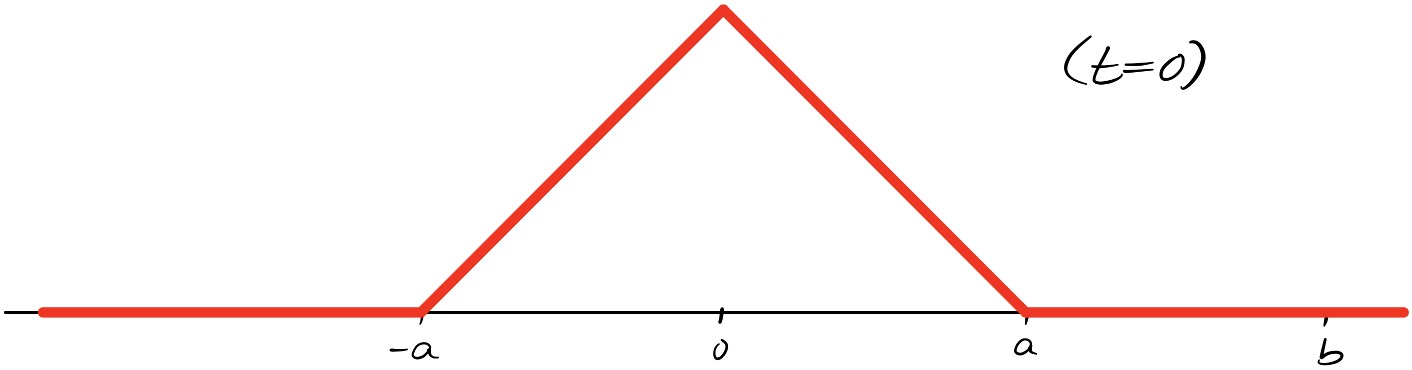
$$E'(t) = \rho\int_{-\infty}^{\infty} u_t\left(\frac{T}{\rho}u_{xx} - r u_t\right)dx + T\int_{-\infty}^{\infty} u_x u_{xt}\,dx$$

$$= T\int_{-\infty}^{\infty} u_t u_{xx}\,dx - \rho\int_{-\infty}^{\infty} r u_t^2\,dx + T\int_{-\infty}^{\infty} u_x u_{xt}\,dx$$

$$= -T\int_{-\infty}^{\infty} \cancel{u_{tx}u_x}\,dx - \rho\int_{-\infty}^{\infty} r u_t^2\,dx + T\int_{-\infty}^{\infty} \cancel{u_x u_{xt}}\,dx$$

$$= -\rho r\int_{-\infty}^{\infty} u_t^2\,dx \leq 0.$$

The last term is non-positive, so we conclude that $E'(t) \leq 0$, i.e. the energy might decrease.



d) i) The disturbance moves at speed $c = \sqrt{\frac{T}{\rho}}$.
Since $\psi = 0$, the solution is simply

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$$

So the disturbance will be felt at $b > a$ at time $t_1 = \frac{b-a}{c} = \sqrt{\frac{\rho}{T}}(b-a)$

ii) The disturbance will stop vibrating at $b > a$ since $\psi = 0$. Once the left edge of the disturbance passes through b, there will be no more vibrations there. This will happen at time:

$$t_2 = \frac{b+a}{c} = \sqrt{\frac{\rho}{T}}(b+a)$$

2. **The Diffusion Equation.** Consider the *diffusion equation* in the interval $(0, \ell)$ with Dirichlet boundary conditions:

$$\begin{cases} u_t(x,t) - ku_{xx}(x,t) = 0, & 0 < x < \ell, \quad t > 0, \\ u(0,t) = u(\ell,t) = 0, & t > 0, \\ u(x,0) = \phi(x), & 0 < x < \ell. \end{cases}$$

Assume that $k > 0$ and that the function ϕ is continuous on $[0, \ell]$, non-negative and not identically 0. Let $T > 0$ and define the rectangle

$$R := [0, \ell] \times [0, T]$$

in the (x,t) plane. Define Γ to be the union of the bottom, right and left edges of R.

(a) State the maximum principle for R. [3]

(b) State the *strong* maximum principle for R. [3]

(c) Use the energy method to prove that $\int_0^\ell u(x,t)^2 \, dx$ is a *strictly* decreasing function of t. *Hint: multiply the equation by u and integrate.* [9]

(d) Separate the variables $u(x,t) = X(x)T(t)$ to express u in series form (you may assume that the equation $-X'' = \lambda X$ with Dirichlet boundary conditions has only positive eigenvalues). [9]

(e) If $\phi(x) = \sin(\frac{2\pi}{\ell}x)$, what are the coefficients in the preceding expansion? (You may use the fact that $\int_0^\ell \sin^2(\frac{2\pi}{\ell}x) \, dx = \frac{\ell}{2}$ and that the eigenfunctions are mutually orthogonal without proof). [9]

a) <u>The maximum principle:</u> The maximum of u in R is obtained on Γ: $\max_R u = \max_\Gamma u$.

b) <u>The strong maximum principle:</u> Unless u is constant, the maximum of u in R is strictly on Γ and <u>not</u> in the interior of R.

c) Multiply the eq. by u to get: $u u_t = k u u_{xx}$.

Integrate: $\quad LHS = \int_0^\ell u u_t \, dx = \frac{1}{2}\frac{d}{dt}\left(\int u^2 dx\right)$.

$RHS = \int_0^\ell k u u_{xx} \, dx = -k\int_0^\ell u_x^2 \, dx + \cancel{\left[k u u_x\right]_{x=0}^{\ell}}$

Since $u(x,0) = \phi(x)$ is continuous, non-negative and not identically 0, and $\phi(0) = \phi(\ell) = 0$, take $R = [0,\ell] \times [0,T]$, and Γ its bottom and sides. So $\min_\Gamma u = 0$, $\max_\Gamma > 0$, so that u must be strictly greater than 0 inside R. Since $u(0,t) = u(\ell,t) = 0$, this implies that u_x cannot be identically 0 along each time slice. Therefore, $RHS < 0$ (strictly!):

$$\frac{d}{dt}\left(\int u^2 dx\right) < 0$$

$\Longrightarrow \int u^2 dx$ is strictly decreasing.

d) $$X(x)T'(t) = kX''(x)T(t) \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda = -\beta^2$$

$$X''(x) + \lambda X(x) = 0 \Rightarrow X(x) = A\cos(\beta x) + B\sin(\beta x)$$

$$X(0) = X(\ell) = 0 \longrightarrow A = 0, \quad \beta_n = \frac{n\pi}{\ell}, \quad X_n = \sin\left(\frac{n\pi}{\ell}x\right)$$

$$T'(t) = -\lambda k T(t) \implies T(t) = Ae^{-\lambda k t}$$

$$\implies u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right) e^{-\left(\frac{n\pi}{\ell}\right)^2 k t}$$

e) $\sin\left(\frac{2\pi}{\ell}x\right) = \phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right)$

$$A_n = \begin{cases} 1 & n=2 \\ 0 & \text{otherwise} \end{cases}$$

$$\implies u(x,t) = \sin\left(\frac{2\pi}{\ell}x\right) e^{-4\left(\frac{\pi}{\ell}\right)^2 k t}$$

3. **The Laplace Equation.**

(a) Let the function u be harmonic in a disk $B \subset \mathbb{R}^2$ of radius $a > 0$ centred at the origin, with $u = h(\theta)$ on ∂B. Poisson's formula is

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi.$$

State and prove the *mean value property.* [5]

(b) Let $D \subset \mathbb{R}^2$ be an open, bounded and connected set. Let the function u be harmonic in D and continuous in $\overline{D} = D \cup \partial D$. State and prove the *strong maximum principle.* [14]

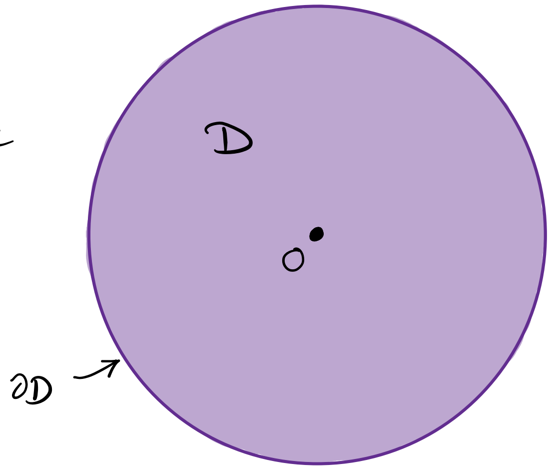
(c) Find the harmonic function $u(x, y)$ in the square

$$R = \{(x, y) \,|\, 0 < x < \pi, 0 < y < \pi\}$$

satisfying the boundary conditions $u(0, y) = u(\pi, y) = u(x, 0) = 0$ and $u(x, \pi) = g(x)$. You may assume that the equation $-X'' = \lambda X$ with Dirichlet boundary conditions has only positive eigenvalues. [14]

Theorem: (Mean Value Property)

Let u be a harmonic function in a disk D and continuous on $\overline{D} = D \cup \partial D$. Then the value of u at the center of D equals the average of u on its circumference ∂D.



Proof: Without loss of generality, assume that the center of D is at $(x,y) = (0,0)$. From Poisson's formula we know that

$$u(r=0) = \frac{a^2}{2\pi}\int_0^{2\pi} \frac{u(\phi)}{a^2}\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} u(\phi)\, d\phi$$

which is, by definition, the average of u on ∂D.

Theorem: (Strong Maximum Principle)

Let D be a connected and bounded open set in $\mathbb{R}^2$. Let $u(x,y)$ be harmonic in D and continuous in $\overline{D} = D \cup \partial D$. Then the max and min of u are attained on ∂D and nowhere inside D (unless u is a constant function).

Suppose the u attains its max M at some point $\vec{p}_M \in D$. Let $\vec{p} \in D$ be any other point. Let Γ be a curve contained in D linking $\vec{p}_M$ and $\vec{p}$. Let $d > 0$ be the distance between Γ and ∂D (d is positive since both $\vec{p}_M, \vec{p}$ are in D, Γ is chosen to be in D and D itself is open).

Let B_1 be a disk centered at $\vec{p}_M$ with radius $\frac{d}{2}$.

Then $B_1 \subset D$. By the mean value property,

$$M = u(\vec{p}_M) = \text{average of } u \text{ on } \partial B_1$$

The average of u on any set cannot exceed M. So we have: $M = \text{average on } \partial B_1 \leq M$. Hence the average must be $= M$. The value of u cannot exceed M at any point on ∂B_1; so, in order for the average to be M, the value of u also cannot be $< M$ at any point. Hence $u = M$ on ∂B_1.

The same argument can be repeated for any disk of radius $\alpha \frac{d}{2}$ around $\vec{p}_M$, for any $\alpha \in (0,1)$. Hence $u = M$ on the entire disk B_1.

Now, choose a point $\vec{p}_1 \in \Gamma \cap \partial B_1$. $u(\vec{p}_1) = M$.

Let B_2 be a disk of radius $\frac{d}{2}$ centered at $\vec{p}_1$. By the same argument as before (applied to $\vec{p}_1$ instead of $\vec{p}_M$), $u \equiv M$ on B_2.

Chose a point $\vec{p}_2 \in \Gamma \cap \partial B_2$ and repeat these arguments.

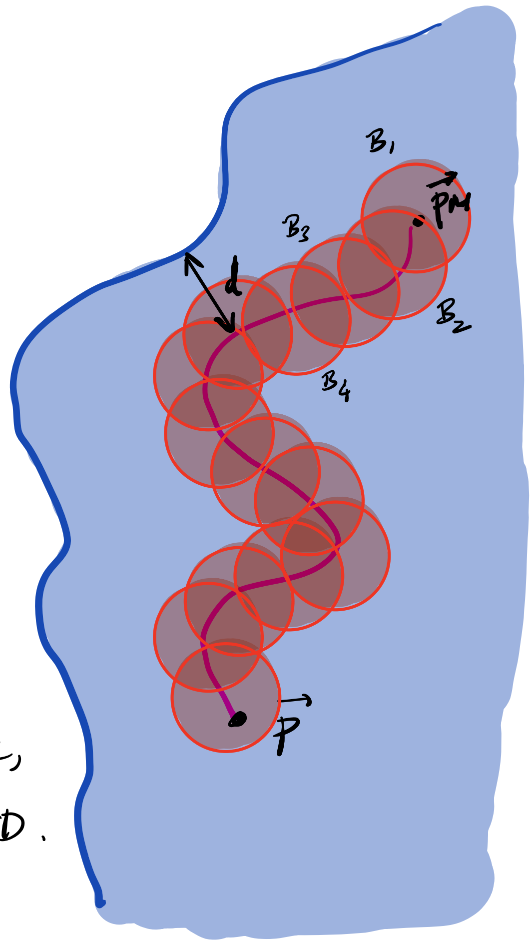
Important point: since Γ is a closed curve, and all disks B_n are of a fixed radius, only finitely many are required to cover Γ.

Conclusion: $u(\vec{p}) = M$.

However, $\vec{p}$ was arbitrary.

So $u = M$ everywhere in D

Conclusion: if the max is attained in D, then u is simply constant. Otherwise, the max would necessarily have to be on ∂D.



4. **Properties of Differential Operators and First-Order PDEs.**

(a) Solve the first-order equation [8]

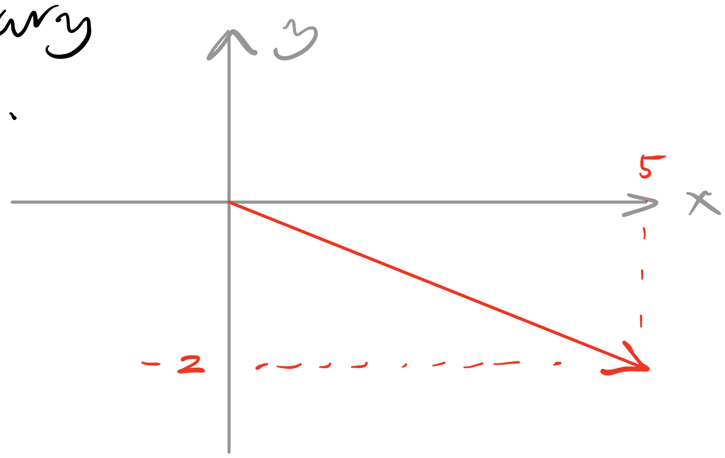
$$\begin{cases} 5u_x(x,y) - 2u_y(x,y) = 0, \\ u(x,0) = \cos x. \end{cases}$$

(b) Let $\mathcal{L}$ be the operator given by $\mathcal{L}f(x) = -f''(x)$ on some interval (a,b) with either Dirichlet, Neumann or Periodic boundary conditions. Prove that $\mathcal{L}$ has only real eigenvalues, and that its eigenfunctions can be taken to be real-valued. In your proof you may use Green's second identity for two twice continuously differentiable functions $y_1(x), y_2(x)$ on (a,b), and continuous on $[a,b]$: [20]

$$\int_a^b (-y_1''\overline{y_2} + y_1\overline{y_2}'')\,dx = (-y_1'\overline{y_2} + y_1\overline{y_2}')\,|_{x=a}^{b}.$$

(c) If $\mathcal{L}$ is subject to Neumann boundary conditions, can 0 be an eigenvalue? Explain your answer. [5]

Example: let $a = 5$, $b = -2$
and consider the auxiliary condition $u(x,0) = \cos x$.



$$5u_x - 2u_y = 0$$

$$u(x,y) = f(-2x - 5y)$$
is the general solution.

$$u(x,0) = \cos x = f(-2x)$$

Substitute $w = -2x \implies f(w) = \cos\left(-\frac{w}{2}\right)$

Hence the solution is: $u(x,y) = \cos\left(x + \frac{5}{2}y\right)$

We can check:

$$5u_x - 2u_y = -5\sin\left(x+\tfrac{5}{2}y\right) + \cancel{2}\sin\left(x+\tfrac{5}{2}y\right)\cdot\frac{5}{\cancel{2}} = 0.$$

Proof: In Green's Second Identity ⊛ replace y_1, y_2 with some function $X(x)$. Then

$$\left(-X'\bar{X} + X\bar{X}'\right)\Big|_{x=a}^{b} = \int_a^b \left(-X''\bar{X} + X\bar{X}''\right) dx$$

Now suppose that X is an eigenfunction of $\mathcal{L}_D, \mathcal{L}_N$ or $\mathcal{L}_P$ with eigenvalue λ.

From the Lemma we know that the LHS $= 0$. Hence:

$$0 = \int_a^b \left(-X''\bar{X} + X\bar{X}''\right) dx = \int_a^b \left(\lambda X\bar{X} - X\lambda^*\bar{X}\right) dx$$
$$= (\lambda - \lambda^*)\int_a^b X(x)\bar{X}(x)\, dx = (\lambda - \lambda^*)\int_a^b |X(x)|^2\, dx$$

Since $|X(x)|^2 \geq 0$ and since $X(x)$ is not trivially 0, the integral $\int_a^b |X(x)|^2 dx$ is strictly positive (why?). Therefore we must have $\lambda - \lambda^* = 0$ which can only be true if $\lambda \in \mathbb{R}$.

We need to show that $X(x)$ can be taken to be real-valued. Suppose that $X(x)$ is complex-valued and write it as $X(x) = Y(x) + i\,Z(x)$ where Y, Z are real-valued. Then:

$$-Y''(x) - i\,Z''(x) = -X''(x) = \lambda X(x) = \lambda Y(x) + i\lambda Z(x)$$

Taking real and imaginary parts we have:

$$-Y''(x) = \lambda Y(x) \qquad -Z''(x) = \lambda Z(x)$$

We know that X satisfies (D), (N), or (P). Y and Z will satisfy the same BCs as well (check this!).

So Y, Z are real-valued eigenfunctions satisfying the same BCs as X. Since $\bar{X}$ has eigenvalue $\lambda^* = \lambda$ (eigenvalues are real!) we conclude that we can replace $X, \bar{X}$ by Y, Z, observing that $\operatorname{span}\{X, \bar{X}\} = \operatorname{span}\{Y, Z\}$.

So we have shown that λ can be taken with eigenfunctions Y and Z (which are both real) rather than $X, \bar{X}$. $\square$