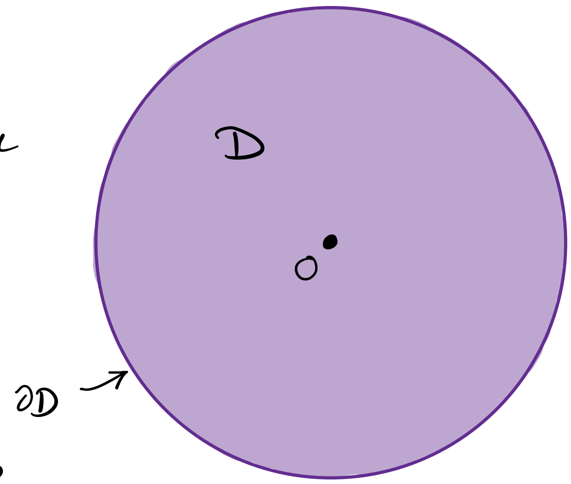
Theorem: (Mean Value Property)

Let u be a harmonic function in a disk D and continuous on $\overline{D} = D \cup \partial D$. Then the value of u at the center of D equals the average of u on its circumference ∂D.



Proof: Without loss of generality, assume that the center of D is at $(x, y) = (0, 0)$.

From Poisson's formula we know that

$$u(r=0) = \frac{a^2}{2\pi} \int_0^{2\pi} \frac{u(\phi)}{a^2}\, d\phi = \frac{1}{2\pi} \int_0^{2\pi} u(\phi)\, d\phi$$

which is, by definition, the average of u on ∂D. $\square$

Theorem: (Strong Maximum Principle)

Let D be a connected and bounded open set in $\mathbb{R}^2$. Let $u(x,y)$ be harmonic in D and continuous in $\overline{D} = D \cup \partial D$. Then the max and min of u are attained on ∂D and nowhere inside D (unless u is a constant function).

Proof: Let $\vec{p}_M \in D$ be such that the max of u is attained there: $u(\vec{p}_M) = \max_{\overline{D}} u$. Let $\vec{p} \in D$ be any other point in D. Since D is connected, there exists a continuous path Γ connecting $\vec{p}_M$ and $\vec{p}$ s.t. $\Gamma \subseteq D$. Since Γ and ∂D are both closed and disjoint, there is a positive distance between them, denote it $d = \text{dist}(\Gamma, \partial D) > 0$.

Let $\overline{B}_1 = \overline{B}(\vec{p}_M, \frac{d}{2})$ be the closed disk of radius $d/2$ around $\vec{p}_M$. Then $\overline{B}_1 \subseteq D$ so that u is harmonic in $\overline{B}_1$. By the mean value property, $u(\vec{p}_M) =$ average of u on ∂B_1.

Since $u(\vec{p}_M) = \max_{\overline{D}} u$, $u(\vec{p}_M) \geq u(\vec{q})$ for any $\vec{q} \in \partial B_1$.

If $\exists \vec{q} \in \partial B_1$ s.t. $u(\vec{q}) < u(\vec{p}_M)$ then necessarily the average on ∂B_1 is $< u(\vec{p}_M)$. Hence $u(\vec{q}) = u(\vec{p}_M)$ $\forall q \in \partial B_1$.

This argument holds for any disk of radius $< d/2$ around $\vec{p}_M$ too, so that $u(\vec{q}) = u(\vec{p}_M)$ for any $\vec{q} \in \overline{B}_1$.

Hence u is constant and $= u(\vec{p}_M)$ on $\overline{B}_1$.

Since Γ is closed and bounded it is compact, so that every cover by open sets admits a finite subcover. Consider the set of all open disks of radius $d/2$ centered at points in Γ. These disks cover Γ and therefore admit a finite subcover. Including B_1 in this cover, we may label the elements of this cover $B_1, B_2, \dots, B_N$, arranged by their order along Γ. Without loss of generality, we may assume that the center of B_{i+1} lies in B_i. Then the center of B_2 lies in B_1. But $u = u(\vec{p}_M)$ in B_1. So by the same argument we had for B_1, $u = u(\vec{p}_M)$ also on B_2. By induction this is true for all B_i, so that $u = u(\vec{p}_M)$ on all of Γ. In particular, $u(\vec{p}) = u(\vec{p}_M)$. But $\vec{p}$ was arbitrary. Therefore, $u \equiv \text{const}$ in D. The only other option is that the max is attained on ∂D.