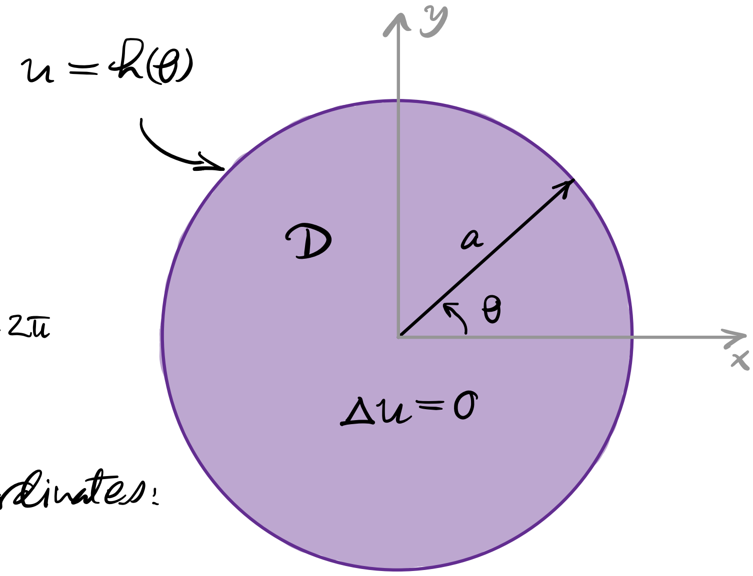
6.3 Poisson's Formula

We consider the problem

$$\begin{cases} \Delta u = 0 & r < a \\ u = h(\theta) & r = a,\ 0 < \theta < 2\pi \end{cases}$$



We separate variables in polar coordinates:

$$u(r,\theta) = R(r)\,\Theta(\theta).$$

The formula for Δ in polar coordinates is (we have seen this)

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 .$$

Hence we get:

$$0 = \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$
$$= R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''.$$

Multiply by r^2
Divide by $R\Theta$ $\Longrightarrow$ $r^2\dfrac{R''}{R} + r\dfrac{R'}{R} + \dfrac{\Theta''}{\Theta} = 0$

$$\Longrightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

So we find the two equations:

$$\Theta''(\theta) + \lambda\,\Theta(\theta) = 0$$
$$r^2 R''(r) + r R'(r) - \lambda R(r) = 0$$

The $\Theta(\theta)$ equation: It is natural to impose *periodic* boundary conditions, so we have:

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(\theta) = \Theta(\theta + 2\pi) \end{cases}$$

From our abstract theorems we know that all eigenvalues are real and non-negative. Verify that:

0 eigenvalue comes with $\Theta_0(\theta) = \text{const}$

$\lambda_n = n^2$ $\Theta_n(\theta) = A\cos(n\theta) + B\sin(n\theta)$.

The $R(r)$ equation: try $R(r) = r^\alpha$ to get:

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0$$

$$\Longrightarrow \quad (\alpha^2 - n^2) r^\alpha = 0$$

$$\Longrightarrow \quad \alpha_n = \pm n$$

$$\Longrightarrow \quad R_n(r) = Cr^n + Dr^{-n}$$

and for $\lambda = 0$ $\quad R_0(r) = C + D\ln r$ (Check this)

Boundary condition at $r=0$: we can't allow functions that are unbounded (r^{-n}, $\ln r$) so we set their coefficients to 0.

$$\longrightarrow \quad \boxed{u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n \left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)}$$

Inhomogeneous boundary condition:

$$h(\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

$$\longrightarrow \quad \boxed{\begin{aligned} A_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos(n\phi)\,d\phi \\ B_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin(n\phi)\,d\phi \end{aligned}}$$

Plug these into the eq for u to get:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)\left[\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi)\right]d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \underbrace{\cos(\theta-\phi)}_{\frac{e^{in(\theta-\phi)} - e^{-in(\theta-\phi)}}{2}}\right]$$

$$1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}$$

geometric series

$$= 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{i(\theta-\phi)}}$$

$$= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}$$

$$\Longrightarrow \quad \boxed{u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\,d\phi}$$

This is called Poisson's formula. Remarkably, it gives a complete characterization of the surface $u(x,y)$ using only the knowledge of the values of h, i.e. the values of u along the boundary of the disk.