6.2 Rectangles and Cubes

In the case of $\Delta u = 0$ in an open set D which is a rectangle (2D) or cube (3D), we can use our favorite method of <u>separation</u> of <u>variables</u> because of the special geometry. Here we only focus on 2D for simplicity. The general approach is as follows:

(i) Separate $u(x,y) = X(x)\,Y(y)$.

(ii) Impose the homogeneous BCs to get the eigenvalues.

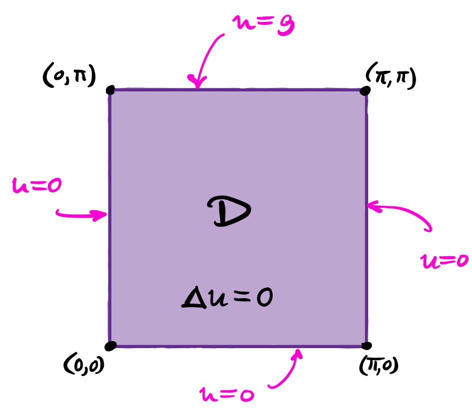
(iii) Sum the series.

(iv) Impose the inhomogeneous conditions.

We see this through an example.

Example: Find the harmonic function $u(x,y)$ in the square $D = \{(x,y) \mid 0<x<\pi,\ 0<y<\pi\}$ with the BCs:

$$u(0,y) = u(\pi,y) = u(x,0) = 0, \qquad u(x,\pi) = g(x).$$



(i) Separate variables: $u(x,y) = X(x)\,Y(y)$.

$$0 = \Delta u = \Delta(XY) = X''(x)\,Y(y) + X(x)\,Y''(y)$$

$$\rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda, \qquad \lambda \in \mathbb{C}$$

$$\rightarrow \quad \begin{aligned} X''(x) &= -\lambda X(x) \qquad 0<x<\pi \\ Y''(y) &= +\lambda Y(y) \qquad 0<y<\pi \end{aligned}$$

(ii) $X(x)$ satisfies: $\begin{cases} X''(x) + \lambda X(x) = 0 & 0<x<\pi \\ X(0) = X(\pi) = 0 \end{cases}$

We already know that this problem (Dirichlet problem) has positive eigenvalues $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 = n^2$ $n=1,2,\ldots$ (since $\ell=\pi$).
with eigenfunctions: $X_n(x) = \sin(nx)$ $n=1,2,\ldots$

$Y(y)$ satisfies: $\begin{cases} Y''(y) - \lambda Y(y) = 0 & 0<y<\pi \\ Y(0) = 0 \end{cases}$ [The inhomogeneous condition $Y(\pi)X(x) = g(x)$ comes later]

We already know that $\lambda = \lambda_n > 0$, so that the eq. $Y'' - \lambda Y = 0$ has solutions of the form $Y_n(y) = A\cosh(\beta_n y) + B\sinh\beta_n y$ where $\beta_n^2 = \lambda_n$. Now we impose $Y(0)=0$: $0 = Y(0) = A$.

Hence $Y_n(y) = \sinh(ny)$ (recalling that $\beta_n = n$ here and taking $B=1$)

(iii) We can now sum the series:

We found: $X_n(x) = \sin(nx)$
$Y_n(y) = \sinh(ny)$

$$\Longrightarrow \quad u(x,y) = \sum_{n=1}^{\infty} A_n \sin(nx)\sinh(ny)$$

(iv) Finally we impose $g(x) = u(x,\pi) = \sum_{n=1}^{\infty} \overbrace{A_n \sinh(n\pi)}^{\tilde{A}_n} \sin(nx)$

This is a Fourier sine series, and we already know that:

$A_n \sinh(n\pi) = \tilde{A}_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx$, so that:

$$A_n = \frac{2}{\pi\sinh(n\pi)}\int_0^{\pi} g(x)\sin(nx)\,dx.$$