## 5.1 The Coefficients of a Fourier Series

As we've seen in the previous section, on $(0,\ell)$ the initial condition $\phi(x)$ was represented as

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right) \quad \text{in the Dirichlet case}$$

Fourier sine series

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right) \quad \text{in the Neumann case.}$$

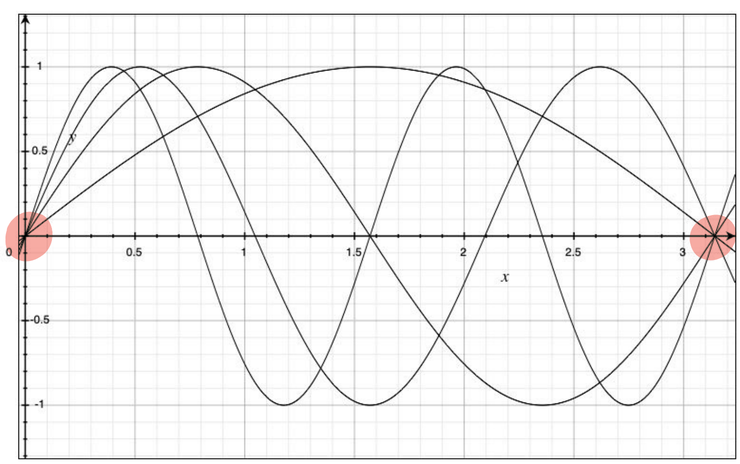
Fourier cosine series

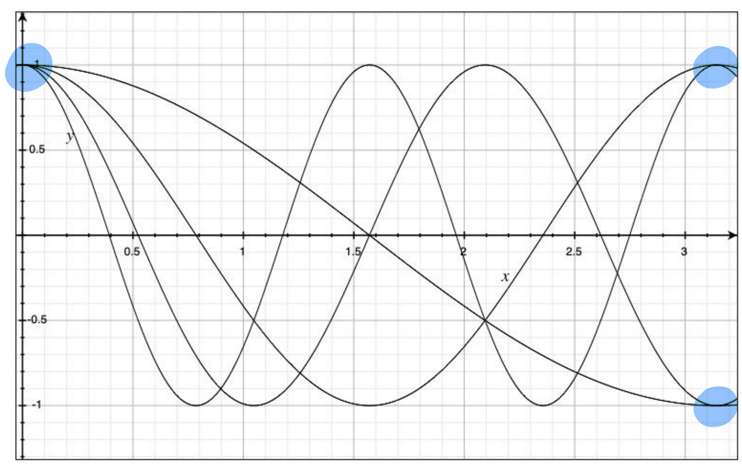
This was true for both the wave and diffusion equations.

It turns out that on $(0,\ell)$ essentially any function can be represented as a Fourier sine series and as a Fourier cosine series.

$\sin\left(\frac{n\pi}{\ell}x\right)$ on $(0,\ell)$ $\qquad\qquad$ $\cos\left(\frac{n\pi}{\ell}x\right)$ on $(0,\ell)$

$n=1,2,3,4$




(here we took $\ell = \pi$)

That is, both the sines and the cosines form a <u>basis</u> for functions on $(0,l)$. However, notice that the sines are <u>always</u> 0 at the endpoints, and the cosines <u>always</u> have a derivative which is 0 at the endpoints.

Compare to $\mathbb{R}^n$: suppose that $\{v_i\}_{i=1}^n$, $\{u_i\}_{i=1}^n$ are two bases for $\mathbb{R}^n$. Then any $v \in \mathbb{R}^n$ can be represented as

$$v = \sum_{i=1}^n a_i v_i \quad \text{and as} \quad v = \sum_{i=1}^n b_i u_i$$

uniquely.

If the basis elements are orthogonal to one another (i.e. $(v_i, v_j) = 0$ and $(u_i, u_j) = 0$ iff $i \neq j$) then we can find the coefficients:

$$(v, v_j) = \left(\sum_{i=1}^n a_i v_i, v_j\right) = \sum_{i=1}^n a_i (v_i, v_j) = a_j \|v_j\|^2$$

$$\longrightarrow \quad a_j = \frac{1}{\|v_j\|^2}(v, v_j).$$

So first we check for orthogonality of these bases. What does orthogonality even mean for functions on $(0,l)$? It means that their joint integral is 0: $f(x), g(x)$ are orthogonal iff $\int_0^l f(x)\,g(x)\,dx = 0$.

Lemma:

$$\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)dx = \begin{cases} 0 & \text{if } n\neq m \\ \frac{\ell}{2} & \text{if } n=m \end{cases}$$

$$\int_0^\ell \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)dx = \begin{cases} 0 & \text{if } n\neq m \\ \frac{\ell}{2} & \text{if } n=m \end{cases}$$

Proof: We show just for the sines. It is identical for the cosines. Suppose that $n\neq m$. Using the identity

$$\sin\alpha\sin\beta = \tfrac{1}{2}\cos(\alpha-\beta) - \tfrac{1}{2}\cos(\alpha+\beta)$$

we have:

$$\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right) = \tfrac{1}{2}\cos\left(\frac{\pi x}{\ell}(n-m)\right) - \tfrac{1}{2}\cos\left(\frac{\pi x}{\ell}(n+m)\right)$$

Integrating $\int_0^\ell$ we find:

$$\int_0^\ell \tfrac{1}{2}\cos\left(\frac{\pi x}{\ell}(n-m)\right)dx = \frac{\ell}{2(n-m)\pi}\sin\left(\frac{\pi x}{\ell}(n-m)\right)\Big|_{x=0}^{\ell}$$
$$= \frac{1}{2(n-m)\pi}\Big[\underbrace{\sin(\pi(n-m))}_{0} - \underbrace{\sin 0}_{0}\Big] = 0$$

For $n=m$ we use $\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$

$$\int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right)dx = \int_0^\ell \left(\tfrac{1}{2} - \tfrac{1}{2}\cos\left(\frac{2n\pi x}{\ell}\right)\right)dx$$
$$= \tfrac{1}{2}\ell - \frac{\ell}{4n\pi}\sin\left(\frac{2n\pi x}{\ell}\right)\Big|_{x=0}^{\ell}$$
$$= \tfrac{1}{2}\ell - \frac{\ell}{4n\pi}\Big[\underbrace{\sin(2n\pi)}_{0} - \underbrace{\sin 0}_{0}\Big] = \tfrac{1}{2}\ell$$

$\square$

**Fourier Sine Series:** Consider again

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right) \qquad \circledast$$

**Proposition:**

We can compute the coefficients $A_n$ and we have:

$$A_n = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

**Proof:** Multiply $\circledast$ by $\sin\left(\frac{m\pi}{\ell}x\right)$ and integrate $\int_0^\ell$ :

$$\begin{aligned}\int_0^\ell \phi(x)\sin\left(\frac{m\pi}{\ell}x\right)dx &= \int_0^\ell \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx \\ &= \sum_{n=1}^{\infty} A_n \int_0^\ell \sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx \\ \text{Using the Lemma} \longrightarrow \quad &= A_m \cdot \tfrac{1}{2}\ell\end{aligned}$$

$$\longrightarrow \quad A_m = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{m\pi}{\ell}x\right)dx \qquad \blacksquare$$

This expression (and the proof) is completely analogous to what we've seen in the finite-dimensional case $(\mathbb{R}^n)$ above:

$$a_j = \frac{1}{\|V_j\|^2}(V, V_j).$$

**Fourier Cosine Series:** Now we consider:

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right)$$

<u>Proposition:</u> The coefficients are given by:

$$A_n = \frac{2}{\ell}\int_0^{\ell} \phi(x)\cos\left(\frac{n\pi}{\ell}x\right)dx$$

(this explains why we put a $\frac{1}{2}$ in front of $A_0$: otherwise the formula for $A_0$ would differ from the formulas for $A_n$ $(n \neq 0)$ by a factor of 2).

<u>Proof:</u> The proof is identical to the previous proof, so we skip. We just prove for $A_0$, which is different:

$$\begin{aligned}
\int_0^{\ell} \phi(x)\cdot 1 \, dx &= \int_0^{\ell} \left[\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right)\right]\cdot 1 \, dx \\
&= \int_0^{\ell} \frac{1}{2}A_0 \, dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{\ell} \cos\left(\frac{n\pi}{\ell}x\right)dx}_{0} \\
&= \frac{\ell}{2}A_0
\end{aligned}$$

$$\longrightarrow \quad A_0 = \frac{2}{\ell}\int_0^{\ell} \phi(x)\,dx.$$

$\square$

## Full Fourier Series: Now we work on $(-l,l)$

It turns out that on the interval $(-l,l)$ we need all the eigenfunctions used both in the Fourier sine series and in the Fourier cosine series. That is, we need

$$\sin\left(\frac{n\pi}{l}x\right) \quad \text{and} \quad \cos\left(\frac{n\pi}{l}x\right)$$

as well as the constant function $1$ (which is the first term in the Fourier cosine series).

So, on $(-l,l)$ we can represent $\phi(x)$ as:

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi}{l}x\right) + B_n \sin\left(\frac{n\pi}{l}x\right)\right)$$

Lemma:

$$\int_{-l}^{l} \cos\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx = 0 \qquad \forall\, n,m = 1,2,\dots$$

$$\int_{-l}^{l} \cos\left(\frac{n\pi}{l}x\right)\cos\left(\frac{m\pi}{l}x\right)dx = \begin{cases} 0 & \text{if } n\neq m \\ l & \text{if } n=m \end{cases}$$

$$\int_{-l}^{l} \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx = \begin{cases} 0 & \text{if } n\neq m \\ l & \text{if } n=m \end{cases}$$

$$\int_{-l}^{l} 1\cdot \sin\left(\frac{n\pi}{l}x\right)dx = \int_{-l}^{l} 1\cdot\cos\left(\frac{n\pi}{l}x\right)dx = 0$$

$$\int_{-l}^{l}\cos^2\left(\frac{n\pi}{l}x\right)dx = \int_{-l}^{l}\sin^2\left(\frac{n\pi}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}1^2\,dx = l$$

The proof is as before, so we skip. This allows us to compute the coefficients:

<u>Proposition</u>: The coefficients of the full Fourier series are given by:

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \phi(x)\cos\left(\frac{n\pi}{\ell}x\right)dx \qquad n = 0, 1, 2, \ldots$$

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \phi(x)\sin\left(\frac{n\pi}{\ell}x\right)dx \qquad n = 1, 2, 3, \ldots$$

These formulas are <u>not</u> identical to the ones we had before, since the interval is now $(-\ell, \ell)$ rather than $(0, \ell)$.
The proof is as before, so we skip it here.

<u>Example</u>: We shall now consider the function $\phi(x) = x$ in 3 different ways:

1) Represent as a Fourier sine series on $(0, \ell)$.
2) Represent as a Fourier cosine series on $(0, \ell)$.
3) Represent as a full Fourier series on $(-\ell, \ell)$.

1) We want to write $x = \phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right).$ in $(0,\ell)$

We know that : $A_n = \frac{2}{\ell}\int_0^\ell x \sin\left(\frac{n\pi}{\ell}x\right) dx$

int. by parts $\longrightarrow$ $= -\frac{2x}{n\pi}\cos\left(\frac{n\pi}{\ell}x\right)\Big|_{x=0}^{\ell} + \frac{2}{n\pi}\int_0^\ell \cos\left(\frac{n\pi}{\ell}x\right) dx$

$$= -\frac{2\ell}{n\pi}\cos(n\pi) + 0 + \frac{2\ell}{n^2\pi^2}\sin\left(\frac{n\pi}{\ell}x\right)\Big|_{x=0}^{\ell}$$

$$= -\frac{2\ell}{n\pi}(-1)^n + \frac{2\ell}{n^2\pi^2}\left[\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin 0}_{0}\right]$$

$$= (-1)^{n+1}\frac{2\ell}{n\pi}$$

$$\Longrightarrow \quad x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell}{n\pi}\sin\left(\frac{n\pi}{\ell}x\right) = \frac{2\ell}{\pi}\left[\sin\left(\frac{\pi}{\ell}x\right) - \frac{1}{2}\sin\left(\frac{2\pi}{\ell}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{\ell}x\right)\cdots\right]$$

2) Now we write $x = \phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{\ell}x\right)$ in $(0,\ell)$

$$A_n = \frac{2}{\ell}\int_0^\ell x\cos\left(\frac{n\pi}{\ell}x\right) dx$$

$$= \frac{2x}{n\pi}\sin\left(\frac{n\pi}{\ell}x\right)\Big|_{x=0}^{\ell} - \frac{2}{n\pi}\int_0^\ell \sin\left(\frac{n\pi}{\ell}x\right) dx$$

$$= 0 + \frac{2\ell}{n^2\pi^2}\cos\left(\frac{n\pi}{\ell}x\right)\Big|_{x=0}^{\ell}$$

$$= \frac{2\ell}{n^2\pi^2}\left(\underbrace{\cos(n\pi)}_{(-1)^n} - \underbrace{\cos 0}_{1}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ -\frac{4\ell}{n^2\pi^2} & n \text{ odd} \end{cases}$$

$$A_0 = \frac{2}{\ell}\int_0^\ell x\, dx = \frac{2}{\ell}\frac{1}{2}x^2\Big|_{x=0}^{\ell} = \ell$$

$$\Rightarrow \quad x = \frac{1}{2}\ell - \sum_{n \text{ odd}}\frac{4\ell}{n^2\pi^2}\cos\left(\frac{n\pi}{\ell}x\right) = \frac{1}{2}\ell - \frac{4\ell}{\pi^2}\left[\cos\left(\frac{\pi}{\ell}x\right) + \frac{1}{9}\cos\left(\frac{3\pi}{\ell}x\right) + \cdots\right]$$

in $(-\ell,\ell)$

3) Finally $x = \phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi}{\ell}x\right) + B_n \sin\left(\frac{n\pi}{\ell}x\right)\right]$

$$A_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} x \, dx = 0$$

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} x \cos\left(\frac{n\pi}{\ell}x\right) dx = \frac{x}{n\pi}\sin\left(\frac{n\pi}{\ell}x\right)\Big|_{x=-\ell}^{\ell} + \frac{\ell}{n^2\pi^2}\cos\left(\frac{n\pi}{\ell}x\right)\Big|_{x=-\ell}^{\ell}$$

$$= \frac{\ell}{n\pi}\underbrace{\sin(n\pi)}_{0} + \frac{\ell}{n\pi}\underbrace{\sin(-n\pi)}_{0} + \frac{\ell}{n^2\pi^2}\left[\underbrace{\cos(n\pi)}_{(-1)^n} - \underbrace{\cos(-n\pi)}_{(-1)^n}\right] = 0$$

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} x \sin\left(\frac{n\pi}{\ell}x\right) dx = -\frac{x}{n\pi}\cos\left(\frac{n\pi}{\ell}x\right)\Big|_{x=-\ell}^{\ell} + \frac{\ell}{n^2\pi^2}\sin\left(\frac{n\pi}{\ell}x\right)\Big|_{x=-\ell}^{\ell}$$

$$= -\frac{\ell}{n\pi}\underbrace{\cos(n\pi)}_{(-1)^n} - \frac{\ell}{n\pi}\underbrace{\cos(-n\pi)}_{(-1)^n} + \frac{\ell}{n^2\pi^2}\left[\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(-n\pi)}_{0}\right]$$

$$= (-1)^{n+1}\frac{2\ell}{n\pi}$$

$$\Rightarrow \quad x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell}{n\pi}\sin\left(\frac{n\pi}{\ell}x\right) = \frac{2\ell}{\pi}\left[\sin\left(\frac{\pi}{\ell}x\right) - \frac{1}{2}\sin\left(\frac{2\pi}{\ell}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{\ell}x\right)\cdots\right]$$

Notice that this is identical to the Fourier sine series that we found before!

We will see why this is not surprising a bit later.

(Hint: $\phi(x) = x$ is an odd function
$\sin\left(\frac{n\pi}{\ell}x\right)$ are odd functions
$\cos\left(\frac{n\pi}{\ell}x\right)$ are even functions )