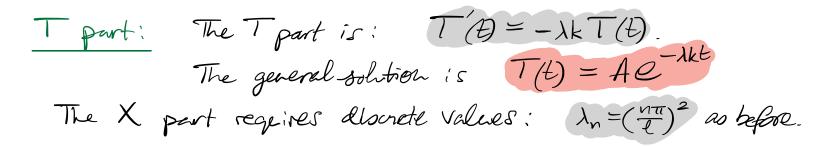
The Diffusion Equation: The analogons problem for the diffusion eq. is:  $u_t(\mathbf{x},t) = \mathbf{k} u_{\mathbf{x}\mathbf{x}}(\mathbf{x},t)$ 0<x<l t>0 n(0,t) = n(l,t) = 0t≥o  $u(x,0) = \phi(x) \qquad 0 < x < l$ Following the same mocedure, we make the ansatz!  $u(\mathbf{x},t)=\mathbf{X}_{(\mathbf{x})}\mathbf{T}(t),$ 

This leads to: X(x)T'(t) = k X''(x)T(t). Dividing by kXT we get

$$-\frac{T'}{kT} = -\frac{x''}{x} = \lambda$$

as before (where we have inserted a - sign since we anticipate  $\lambda$  to be possitive). Notice that the temporal part only has a T', not T": this is going to be crucial!

 $\frac{X \text{ part:}}{(X)} \text{ fts before, we have } -X''(X) = \lambda X(X)$ with the boundary conductions X(0) = X(l) = 0. Exactly as before, we get multiples of  $\sin(\frac{n\pi}{l}X)$ .



So we find the sequence of solutions:  

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{t}\right)^2 kt} sin\left(\frac{n\pi}{t}x\right)$$

And the general solution is any 
$$(finite)$$
 sem:  
 $u(x,t) = \sum_{n}^{\infty} A_n e^{-\binom{m\pi}{e}^2 kt} sin(\frac{m\pi}{e}x)$ 

and to satisfy the initial condition we read:  

$$\phi(x) = u(x, 0) = \sum_{n}^{\infty} A_n e^{-0} \sin(\frac{n\pi}{\ell} x) = \sum_{n}^{\infty} A_n \sin(\frac{n\pi}{\ell} x)$$

Comparing temporal behavior:

The temporal parts of the solutions contain:

• 
$$A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)$$
 WAVE  
 $-\left(\frac{n\pi}{l}\right)^2 kt$   
•  $A_n e$  DIFFUSION

This again demonstrates that solutions to the wave eq. conserve energy (sin and cos are periodic) while solutions to the diffusion eq. have decaying energy (deceasing exponential). Fourier Serves:

The expression of a function as a surr of sines and cosines is called a tourier series (Joseph Fourier 1768-1830),

It turns out that we can take infinite sums (but with courtion!),

This is an extremely important concept in all the sciences.

This can be done for almost any function.

When we wrote  $\phi(x) = \sum_{n} A_n \sin\left(\frac{n\pi}{l}x\right)$ , then the EHS is called the Fourter expansion of  $\phi$ . Since there are only sines, the series is called a Fourier sine series.

Below well have an example with a series of casines. That will be called a Fourier casine series. Comparison to Linear Algebra:

Let's look again at the equation  $X''(x) = -\lambda X(x)$ . Define X to be the operator that seeds a function to its second derivative :  $\mathcal{R}(X(x)) = -X''(x)$ . Then this equation becomes:

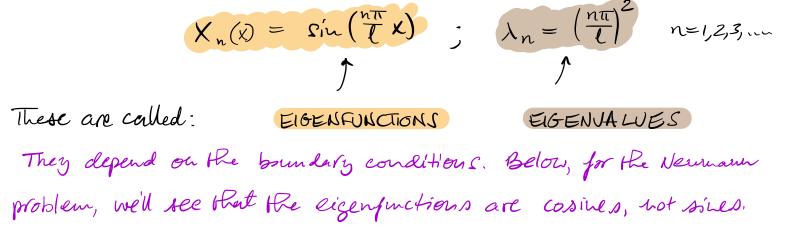
 $\mathcal{L} \times = \lambda \times$ 

Compare this to  $Ax = \lambda x$  which you have seen in linear algebra (where A is an nxn matrix and x is a vector)

This is the same!

In linear algebra, a solution x is called an eigenvector and the corresponding number & is called an eigenvalue.

Here, we discovered that there are infinitely many solutions



Let us justify the assumption that all eigenvalues in are positive numbers:

Proposition: All eigenvalues of 
$$\int -X''(x) = \lambda X(x) \quad 0 < x < l$$
  
 $\lambda X(0) = X(l) = 0$ 

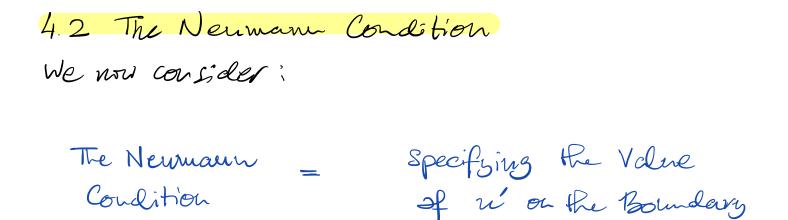
• Could 
$$\lambda = 0$$
 be an eigenvalue?  
Suppose so. Then we would have  $\chi''_{(X)} = 0$   
 $\longrightarrow \chi(x) = D_{X+C}$  for some constants  $C, D$ .  
Plugging in  $\chi(0) = 0 \implies C = 0$   
Then  $\chi(l) = 0 \implies Dl = 0 \implies D = 0$ ,  
So  $\lambda = 0$  cannot be an eigenvalue

• Could 
$$\lambda < 0$$
 be an eigenvalue?  
Suppose so. The write it as  $\lambda = -r^2$  for some  
 $r \in \mathbb{R}$ . Then  $\chi'' = r^2 \chi$   
 $\longrightarrow \chi(\chi) = C \cosh(r\chi) + D \sinh(r\chi)$ .  
 $0 = \chi(0) = C$ ,  $0 = \chi(l) = D \sinh(rl)$ . Since  
 $\chi l \neq 0$  we have  $D = 0$ .  
So  $\lambda < 0$  convot be an eigenvalue.

Could 
$$\lambda$$
 complex be an eigenvalue?  
Suppose so. Denote  $\sqrt{-\lambda} = \pm \gamma$ , where  $\gamma \in C$ .  
The equation  $\chi''(x) = -\lambda \chi(x) = \gamma^2 \chi(x)$  has  
solutions of the form  $\chi(x) = Ce^{\gamma \chi} + De^{-\gamma \chi}$   
(where this is the complex exponential function).  
 $0 = \chi(0) = C + D \implies C = -D$   
 $0 = \chi(l) = Ce^{\gamma l} + De^{-\gamma l} = -De^{\gamma l} + De^{-\gamma l}$   
 $= D(e^{-\gamma l} - e^{\gamma l})$   
 $\implies e^{-\gamma l} - e^{\gamma l} = 0 \implies e^{-\gamma l} = e^{\gamma l}$   
 $\implies e^{2\gamma l} = 1$ 

By properties of the complex exponential function this  
implies that 
$$2\gamma l$$
 is a purely complex number  
and that  $\operatorname{Im}(2\gamma l) = 2\pi n$ ,  $n \in \mathbb{Z}$ .  
 $\Longrightarrow 2\pi\gamma i = 2\pi n$ ,  $n \in \mathbb{Z} \implies \gamma = -i \frac{n\pi}{l}$ ,  $n \in \mathbb{Z}$   
 $\Longrightarrow \lambda = -\gamma^2 = \left(\frac{n\pi}{l}\right)^2$ . But this is real and positive!

So the only eigenches are the positive numbers 
$$\left(\frac{\pi}{l}\right)^2$$
,  $\left(\frac{2\pi}{l}\right)^2$ ,  $\left(\frac{3\pi}{l}\right)^2$ , ....



$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & o < x < l \ t > 0 \\ u_{x}(0,t) = u_{x}(l,t) = 0 & t > 0 \\ u_{x}(0,t) = \phi(x) & u_{t}(x,0) = \psi(x) & o < x < l \end{cases}$$

or the diffusion equation:

$$\begin{cases} u_t(\mathbf{x},t) = \mathbf{k} u_{\mathbf{x}\mathbf{x}}(\mathbf{x},t) & o < \mathbf{x} < l \ t > 0 \\ u_t(\mathbf{x},t) = u_t(l,t) = 0 & t \ge 0 \\ u_t(\mathbf{x},0) = \phi(\mathbf{x}) & o < \mathbf{x} < l \end{cases}$$

Notice that now the boundary conditions involve my rather than n!

Using separation of variables n(x,t) = X(x)T(t) we reach the same equations for X and T as before.

X part: As before, we have  $X''(x) + \beta^2 X(x) = 0$ which leads to solutions of the form:

 $X(x) = C \cos(\beta x) + D \sin(\beta x)$ 

Let's write the derivative of this, which we will need

 $X'(x) = -C_{\beta} S' u(\beta x) + D_{\beta} cod(\beta x)$ 

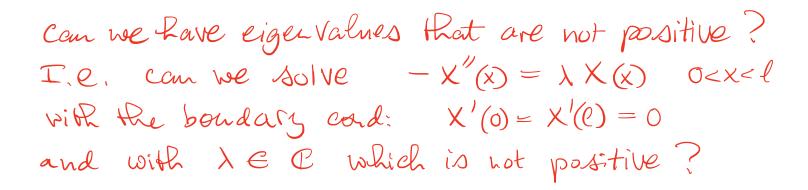
 $u_{\mathbf{x}}(0,t) = 0 \implies \mathbf{x}'(0) = 0 \implies -C_{\mathbf{p}} \underbrace{\sin 0}_{0} + D_{\mathbf{p}} \underbrace{\cos 0}_{1} = 0$  $\implies \mathbf{D} = \mathbf{0},$ 

 $u_{x}(l,t)=0 \implies \chi'(l)=0 \implies -Cp\sin(pl)=0$ 

$\beta l = n\pi$	$\rightarrow$	$\beta_n = \frac{n\pi}{\ell}$	
		$\lambda_n = \left(\frac{\eta \pi}{\ell}\right)^2$	n=1,2,

		$/n\pi$	
$X_n(x) =$	COS	$\left( \overline{\rho} \right)$	$\times)$
		$\sim c$	

These are the EIGENFUNCTIONS for the Neumann problem



$$Tr_{3} \lambda = 0: \quad \text{we get } X'(x) = 0 \quad \text{so that} \\ X(x) = C + Dx, \quad X'_{(w)} = D \\ \text{Apply } BCs: \quad 0 = X'(v) = X'(t) = D. \\ \longrightarrow \quad \text{we can satisfy the } BCs \quad \text{with } D = 0. \\ \longrightarrow \quad X(x) = C \qquad (\text{constant}) \\ \text{is a legitimete solution } (x) \\ \longrightarrow \quad \lambda = 0 \quad is \quad \text{an eigenvalue } (x) \\ \text{is } x = 0 \quad is \quad \text{an eigenvalue } (x) \\ \text{is } x = 0 \quad is \quad \text{an eigenvalue } (x) \\ \text{we can satisfy the eigenvalue$$

 $\lambda < 0$  or  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ : It can be shown that such values of  $\lambda$  cannot be eigenvalues but we skip that for now.

So the eigenvalues are:  $n = 0, 1, 2, \dots$  $\lambda_n = \left(\frac{n\pi}{\varrho}\right)^2$ 

These are the EIGENVALUES for the Neumann problem

Tpart: The T(t) part will be identical to what we saw before, with the exception of the part cowing from 
$$\lambda = 0$$
.

For  $\lambda_n \neq 0$  we again Diffusion equation: have:  $T'(t) = -\lambda_n k T(t)$ 

$$T'(t) = -\lambda_n k T(t)$$

$$\rightarrow T(t) = A e^{-\lambda_n k t}$$

For 
$$\lambda = 0$$
 we have  $T'(\underline{t}) = Ae^{-\lambda_n k t}$   
For  $\lambda = 0$  we have  $T'(\underline{t}) = 0 \implies T(\underline{t}) = A$ .  
So, for  $n = 1, 2, 3, ...$  we have as before:  
 $u_n(\underline{x}, \underline{t}) = A_n e^{-(\underline{n}\underline{t})^2 k t} \cos((\underline{n}\underline{t}, \underline{x}) - n = 1, 2, ...)$ 

Notice that the sine is now a costre!

And we also have a us now: the spatial part is los ("#x) =1 and the temporal port is a constant which we called A above. For reasons which well become clear, we call Ao = 2A, to find:  $u(x,t) = \frac{1}{2}A_0 + \frac{2}{n}A_n e^{-\left(\frac{h\pi}{l}\right)^2 kt} \cos\left(\frac{h\pi}{l}x\right)$ 

In addition, the initial condition will have to satisfy:

$$\phi(x) = h(x, 0) = \frac{1}{2}A_0 + \frac{2}{n}A_n \cos(\frac{n\pi}{t}x)$$

Wave gration: For 
$$\lambda > 0$$
 we get the same  
behavior as we've seen before, so we have:

$$\mathcal{N}_{n}(\mathbf{x},t) = \left[A_{n} \cos\left(\frac{n\pi}{l}ct\right) + B_{n} \sin\left(\frac{n\pi}{l}ct\right)\right] \cos\left(\frac{n\pi}{l}\mathbf{x}\right)$$

For 
$$\lambda = 0$$
 we get  $X_0(x) = const$  as before. For the T  
part we lowe  $T''(t) = \lambda c^2 T(t) = 0$  s that  
 $T_0(t) = A + Bt$ . This To term goes with the  
Xo term which is a constant. So, to conclude,  
the general solution has the Jorn:

 $\mathcal{N}(\mathbf{x},t) = \frac{1}{2}\mathbf{A}_{0} + \frac{1}{2}\mathbf{B}_{0}t + \sum_{n} \left[\mathbf{A}_{n}\cos\left(\frac{n\pi}{\ell}ct\right) + \mathbf{B}_{n}\sin\left(\frac{n\pi}{\ell}ct\right)\right]\cos\left(\frac{n\pi}{\ell}\mathbf{x}\right)$ 

 $\Phi(X) = \mathcal{U}(X, 0) = \frac{1}{2}A_0 + \sum_n A_n \cos\left(\frac{n\pi}{l}X\right)$ 

 $\Psi(\mathbf{x}) = \mathcal{U}_{\mathsf{f}}(\mathbf{x}, \mathbf{0}) = \frac{1}{2} \mathcal{B}_{\mathsf{o}} + \sum_{n} \frac{n\pi}{\mathfrak{f}} \mathcal{C} \mathcal{B}_{n} \cos\left(\frac{n\pi}{\mathfrak{f}} \mathbf{x}\right)$