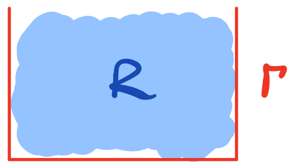
# Detailed Proof of $L^\infty$ - closeness:

**Facts:** 1) For any number $q \in \mathbb{R}$, $|q| = \max(q, -q)$

2) For a cont. function $f(x)$ on $[x_0, x_1]$,

$$\min_{x \in [x_0,x_1]} f(x) = -\max_{x\in[x_0,x_1]} (-f(x))$$

**Max Principle:** we saw that $\max_{\Gamma} u = \max_{R} u$




**Min Principle:** to get the min principle, we apply the max principle to $-u$: $\max_{\Gamma} (-u) = \max_{R} (-u)$

Using Fact 2 above, this implies: $\min_{\Gamma} u = \min_{R} u.$

**$L^\infty$ - closeness:** Consider the problems

$$\begin{cases} u_t - k u_{xx} = f(x,t) & x \in (x_0, x_1) \quad t > t_0 \\ u(x_0,t) = g(t) \quad u(x_1,t) = h(t) & t > t_0 \\ u(x, t_0) = \phi_1(x) & x \in (x_0, x_1) \end{cases}$$

$$\begin{cases} u_t - k u_{xx} = f(x,t) & x \in (x_0, x_1) \quad t > t_0 \\ u(x_0,t) = g(t) \quad u(x_1,t) = h(t) & t > t_0 \\ u(x, t_0) = \phi_2(x) & x \in (x_0, x_1) \end{cases}$$

Suppose that these have solutions $u_1$ and $u_2$ respectively.
Then $w = u_1 - u_2$ satisfies:

$$\begin{cases} w_t - k w_{xx} = 0 \\ w(x_0,t) = w(x_1,t) = 0 \\ w(x, t_0) = \phi_1(x) - \phi_2(x) \end{cases}$$

Max principle implies:

$$w(x,t) \leq \max_{R} w = \max_{\Gamma} w = \max\left\{ \max_{x\in[x_0,x_1]} (\phi_1(x) - \phi_2(x)),\ 0\right\}$$

$$\leq \max_{x\in[x_0,x_1]} |\phi_1(x) - \phi_2(x)|$$

Min principle implies:

$$w(x,t) \geq \min_{R} w = \min_{\Gamma} w = \min\left\{ \min_{x\in[x_0,x_1]} (\phi_1 - \phi_2),\ 0\right\}$$

$$= \min\left\{ -\max_{x\in[x_0,x_1]} (-(\phi_1 - \phi_2)),\ 0\right\}$$

$$= \min\left\{ -\max_{x\in[x_0,x_1]} (\phi_2(x) - \phi_1(x)),\ 0\right\}$$

$$\geq -\max_{x\in[x_0,x_1]} |\phi_2(x) - \phi_1(x)|$$

$$= -\max_{x\in[x_0,x_1]} |\phi_1(x) - \phi_2(x)|$$

So we find that $-\max_{x\in[x_0,x_1]} |\phi_1(x)-\phi_2(x)| \leq w(x,t) \leq \max_{x\in[x_0,x_1]} |\phi_1(x)-\phi_2(x)|$

Denote $M = \max_{x\in[x_0,x_1]} |\phi_1(x) - \phi_2(x)|$ which is necessarily $\geq 0$.

Then:

$$-M \leq w(x,t) \leq M$$

Hence $|w(x,t)| \leq M = \max_{x\in[x_0,x_1]} |\phi_1(x) - \phi_2(x)|$

This is true for every $x \in [x_0,x_1]$, so we can take the max on the LHS:

$$\max_{x\in[x_0,x_1]} |u_1(x,t) - u_2(x,t)| = \max_{x\in[x_0,x_1]} |w(x,t)| \leq \max_{x\in[x_0,x_1]} |\phi_1(x) - \phi_2(x)|$$