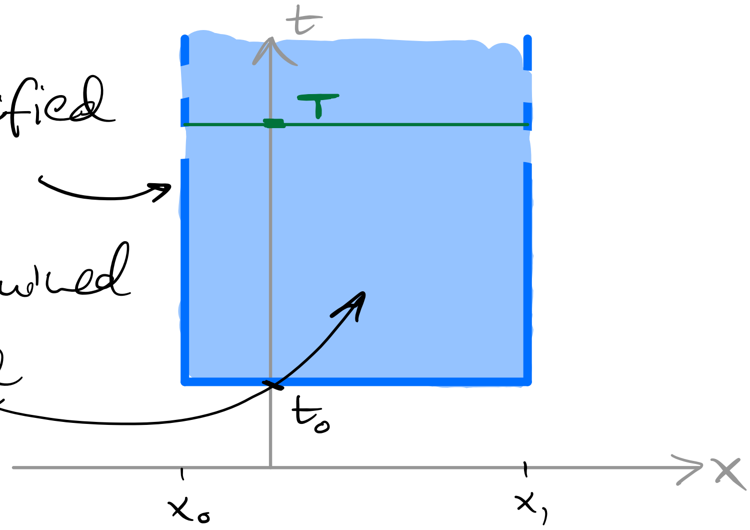
Theorem: (Uniqueness of Solutions)

The Dirichlet problem for the diffusion equation has a unique solution in any rectangle.

That is, if u is specified on the blue edges it is uniquely determined inside the shaded region



The mathematical statement is this:

There is <u>at most</u> one solution to

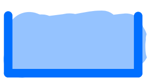
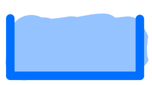
$$\begin{cases} u_t - k u_{xx} = f(x,t) & x_0 \leq x \leq x_1 \quad t \geq t_0 \\ u(x, t_0) = \phi(x) & x_0 \leq x \leq x_1 \\ u(x_0, t) = g(t) \quad u(x_1, t) = h(t) & t \geq t_0 \end{cases}$$

where f, ϕ, g, h are given functions.

<u>Proof</u> 1: We assume that there are two solutions u_1, u_2 and we'll show that they are the same, by showing that $u_1 - u_2$ is identically 0.

Since u_1 and u_2 have the same boundary and initial conditions, their difference $w = u_1 - u_2$ has homogeneous Dirichlet boundary + initial conditions, i.e. 0:

$$\begin{cases} w_t - k w_{xx} = 0 & x_0 \leq x \leq x_1 \quad t \geq t_0 \\ w(x, t_0) = 0 & x_0 \leq x \leq x_1 \\ w(x_0, t) = w(x_1, t) = 0 & t \geq t_0 \end{cases}$$

By the maximum principle the max of w on  is $\leq$ the max of w on  where the top edge of this rectangle can be taken at any $t > t_0$, say $t = T$. But $w = 0$ on , so that $w \leq 0$ on .
Because of the Remark above, w also attains its min on , but this is again 0.
So w is 0 everywhere in . $\blacksquare$

Proof 2: Define w as before. Then

$$w_t - k w_{xx} = 0.$$

Multiply this by w: ← This strategy is called the ENERGY METHOD

$$0 = w \cdot w_t - k w \cdot w_{xx}$$
$$= \frac{1}{2}(w^2)_t - k(w_x w)_x + k w_x^2$$

Integrate in x to get:

$$0 = \int_{x_0}^{x_1} \left[\frac{1}{2}(w(x,t)^2)_t - k\left(w_x(x,t)w(x,t)\right)_x + k w_x(x,t)^2 \right] dx$$
$$= \underbrace{\int_{x_0}^{x_1} \frac{1}{2}(w(x,t)^2)_t \, dx}_{= \frac{d}{dt}\int_{x_0}^{x_1} \frac{1}{2} w(x,t)^2 dx} - \underbrace{k\left[w_x(x,t)w(x,t)\right]_{x=x_0}^{x_1}}_{= k\left[w_x(x_1,t)w(x_1,t) - w_x(x_0,t)w(x_0,t)\right]} + k\int_{x_0}^{x_1} w_x(x,t)^2 dx$$

$w(x_1,t) = 0$, $w(x_0,t) = 0$

So this term is 0

So we have: $$0 = \frac{d}{dt}\int_{x_0}^{x_1} \frac{1}{2} w(x,t)^2 dx + k\int_{x_0}^{x_1} w_x(x,t)^2 dx$$

Hence: $$\frac{d}{dt}\int_{x_0}^{x_1} \frac{1}{2} w(x,t)^2 dx = -k\int_{x_0}^{x_1} w_x(x,t)^2 dx \leq 0$$

So $\int_{x_0}^{x_1} \frac{1}{2} w(x,t)^2 dx$ is a decreasing function of time.

But $\int_{x_0}^{x_1} \frac{1}{2} w(x,t_0)^2 dx = \int_{x_0}^{x_1} \frac{1}{2} \cdot 0^2 dx = 0.$

So $w \equiv 0$ identically. $\blacksquare$

Stability of solutions: In addition to uniqueness, we can show another intuitive aspect of solutions: stability. That is, solutions that are "close" initially, remain "close" at later times.

We now consider:
$$\begin{cases} u_t - k u_{xx} = 0 & x_0 < x < x_1 \quad t > t_0 \\ u(x_0,t) = u(x_1,t) = 0 & t > t_0 \end{cases}$$
and want to compare u_1, u_2 that have ϕ_1, ϕ_2 initially.

Theorem: (L^2-closeness)

For any $t > t_0$,
$$\int_{x_0}^{x_1} [u_1(x,t) - u_2(x,t)]^2 \leq \int_{x_0}^{x_1} [\phi_1(x) - \phi_2(x)]^2 \, dx.$$

So, if this integral is small initially, then this integral is small for all later times.

Proof: From the energy method we saw that $w = u_1 - u_2$ satisfies that $\int_{x_0}^{x_1} w(x,t)^2 dx$ is a decreasing function of time. In particular:

$$\int_{x_0}^{x_1} [u_1(x,t) - u_2(x,t)]^2 = \int_{x_0}^{x_1} w(x,t)^2 \, dx$$
$$\leq \int_{x_0}^{x_1} w(x,t_0)^2 dx = \int_{x_0}^{x_1} [\phi_1(x) - \phi_2(x)]^2 \, dx.$$

Theorem: (L^∞ - closeness, or "uniform" closeness)

For any $t > t_0$

$$\max_{x \in (x_0, x_1)} |u_1(x,t) - u_2(x,t)| \leq \max_{x \in (x_0, x_1)} |\phi_1(x) - \phi_2(x)|$$

Proof: From the maximum principle

$$u_1(x,t) - u_2(x,t) \leq \max_{x \in (x_0, x_1)} |\phi_1(x) - \phi_2(x)|$$

and from the minimum principle:

$$u_1(x,t) - u_2(x,t) \geq - \max_{x \in (x_0, x_1)} |\phi_1(x) - \phi_2(x)|$$

Hence: $$\max_{x \in (x_0, x_1)} |u_1(x,t) - u_2(x,t)| \leq \max_{x \in (x_0, x_1)} |\phi_1(x) - \phi_2(x)|.$$

$\blacksquare$