2.2 Causality and Energy

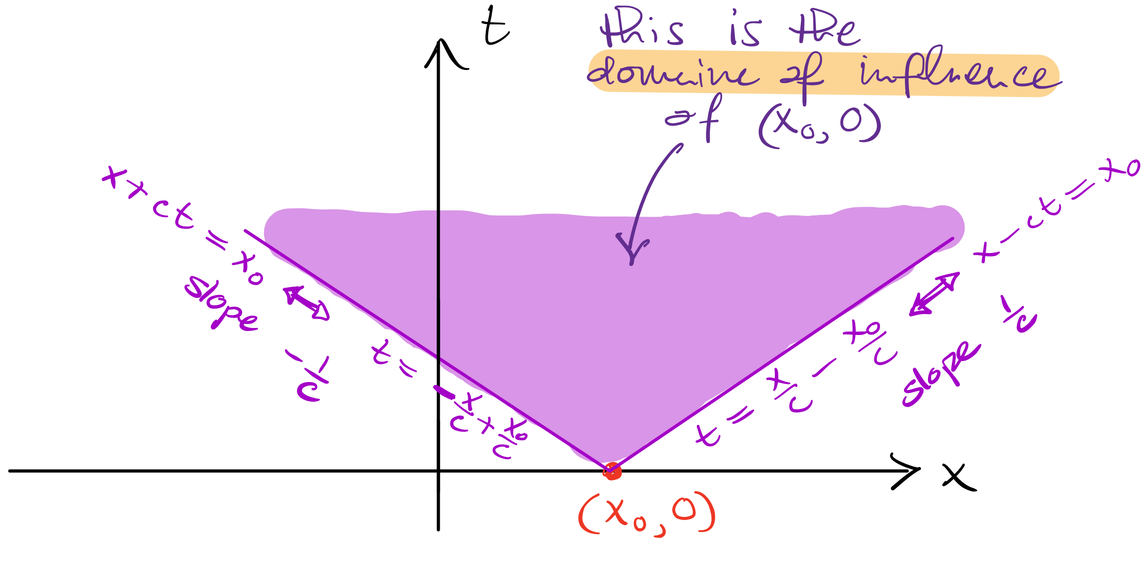
As we have seen, the wave equation in 1D is the sum of two disturbaces: one moving to the right and the other moving to the left.

$\psi \equiv 0 \implies$ signals move at exactly c.

$\psi \not\equiv 0 \implies$ signals move at $\leq c$.

(we haven't proven this, but it can be seen by looking at d'Alembert's formula, and, thinking...)

We usually sketch time-dependent problems with t (time) being the vertical axis, and the horizontal axis representing the spatial coordinate(s) (for now we're in 1D, so there's just x; but in principle there can be $x, y, z \ldots$)

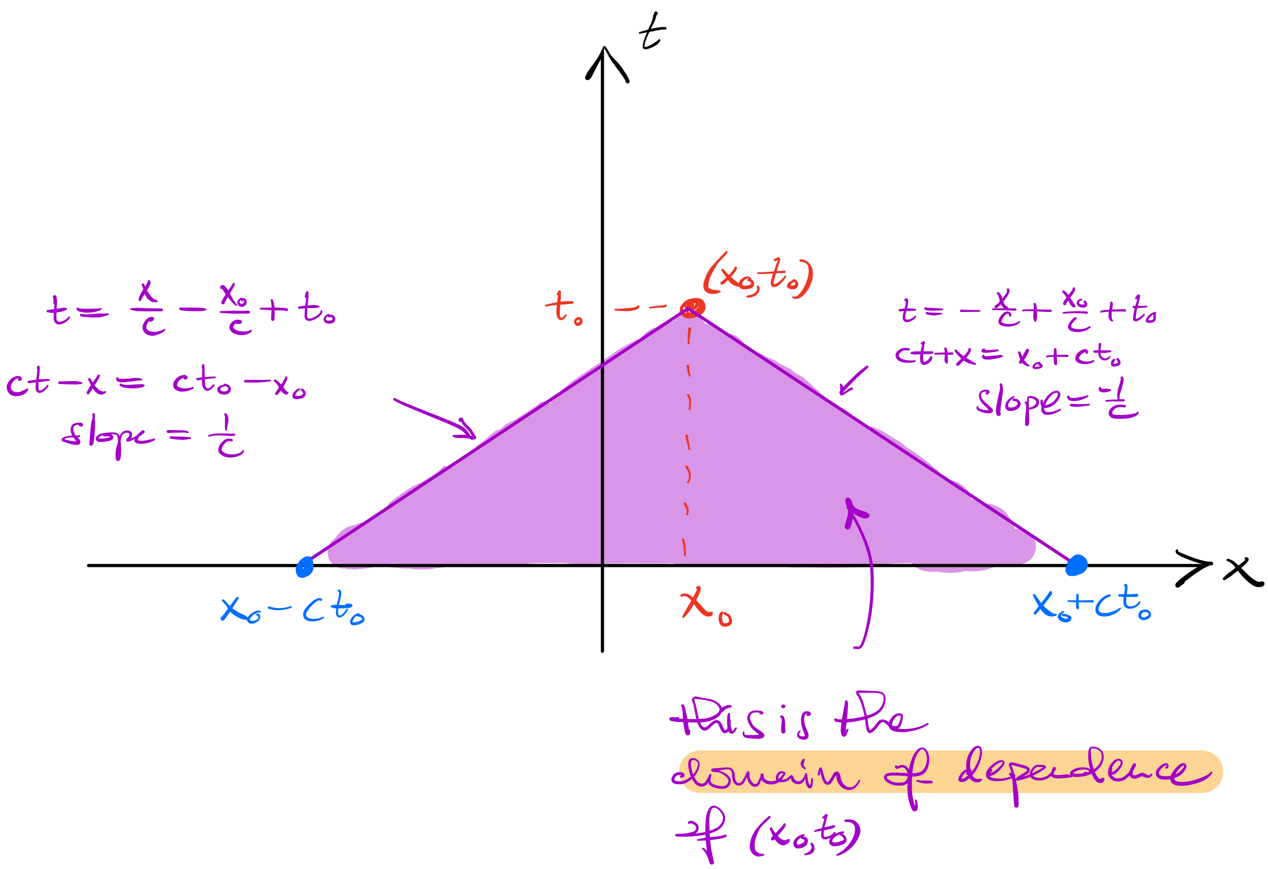


Equally, we can start from some point (x_0, t_0) in space-time, and ask:

What is the past history of (x_0, t_0)?

That is:

Take some time $s < t_0$. Then what are all the points $y \in \mathbb{R}$ s.t. a signal leaving y at time s will reach x_0 at time t_0?



Energy: There is a natural way to associate a non-negative quantity we call "energy" to the wave equation on $\mathbb{R}$. We denote the total energy E. It is made up of two parts:

Kinetic energy: this comes from the actual motion of the string, and is given by

$$E_{kin}(t) = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2 \, dx$$

Potential energy: this is the energy stored in the string, for example when it is stretched.

$$E_{pot}(t) = \frac{1}{2} T \int_{-\infty}^{\infty} u_x^2 \, dx$$

We define: $$E(t) = E_{kin}(t) + E_{pot}(t)$$

Proposition: For a string satisfying the homogeneous wave eq. the energy is conserved: $E'(t) = 0$.

Proof:

$$E'(t) = E_{kin}'(t) + E_{pot}'(t)$$

$$= \rho \int_{-\infty}^{\infty} u_t u_{tt} \, dx + T \int_{-\infty}^{\infty} u_x u_{xt} \, dx$$

$\rho u_{tt} = T u_{xx}$

$$= T \int_{-\infty}^{\infty} u_t u_{xx} \, dx + T \int_{-\infty}^{\infty} u_x u_{xt} \, dx$$

integration by parts of the first term

$$= -T \int_{-\infty}^{\infty} u_{tx} u_x \, dx + \left[T\, u_t u_x \right]_{x=-\infty}^{\infty} + T \int_{-\infty}^{\infty} u_x u_{xt} \, dx$$

The first and last terms cancel out, and the middle term is also 0 since u and its derivatives are assumed to vanish "at" $\pm\infty$. Therefore:

$$E'(t) = 0 \longrightarrow E(t) = E(0) = \text{const.}$$

$$\blacksquare$$

For the plucked string example from before:

$$E = E(0) = \frac{1}{2} T \int_{-\infty}^{\infty} \phi_x^2 \, dx = \frac{1}{2} T \underbrace{\left(\frac{b}{a}\right)^2}_{(\text{slope})^2} \underbrace{2a}_{\substack{\text{length} \\ \text{of interval}}} = \frac{Tb^2}{a}.$$