1.5 Well-Posed Problems

"Well-posedness" is an important concept for identifying "good" problems. It is the following set of "sensible" requirements:

(i) Existence: $\exists$ at least one solution

(ii) Uniqueness: There is at most one solution.

(iii) Stability: A small change in the data, results in a small change in the solution.

(i.e. no "butterfly effect")

The first two conditions are so obvious we might not even think of them. The third condition is very important in applications:

- In real life we rarely know any data precisely. We usually have some errors in our reading of the data. However, if we know that the solution is stable this shouldn't matter too much.
- When we perform computations on a computer there are always rounding errors and other errors coming from approximations. It is important to know that these don't matter too much.

The Backwards Diffusion Equation

(This example can be found in section 2.5 of the book)

Consider the function:

$$u_n(x,t) = \frac{1}{n} \sin(nx)\, e^{-n^2kt}$$

Then:

$$\partial_t u_n = -n^2 k \frac{1}{n} \sin(nx)\, e^{-n^2kt}$$
$$= -kn\sin(nx)\, e^{-n^2kt}$$
$$\partial_x u_n = \frac{n}{n} \cos(nx)\, e^{-n^2kt} = \cos(nx)\, e^{-n^2kt}$$
$$\partial_{xx} u_n = -n \sin(nx)\, e^{-n^2kt}$$

So that $\partial_t u_n = k\,\partial_{xx} u_n$,

i.e. u_n satisfies the diffusion eq.

Suppose we try to start from $t=0$ and go backward in time to $t<0$. Taking $t=-1$ for example, we have

$$u_n(x,-1) = \frac{1}{n} \sin(nx)\, e^{n^2k}$$

Then for n very large or very large negative, $u_n(x,-1)$ will have values of x for which it is extremely large. This violates the <u>stability</u> requirement.

This represents the fact that diffusion has an "arrow of time": it cannot go backward in time. Since much of our universe is comprised of phenomena that obey the diffusion eq. (or variants of it), our world has an "arrow of time" as well, and time cannot be reversed.