1.2 First-order linear equations

The simplest PDEs have the form

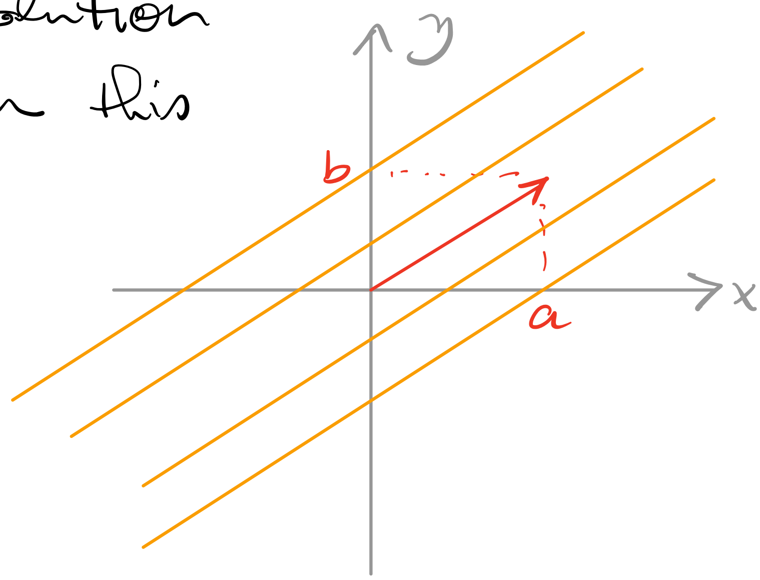
$$au_x + bu_y = 0.$$

a and b can be constant or can depend on the point (x,y). We start with the constant case:

<u>Constant coefficient case:</u>

Notice that $$au_x + bu_y = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)u = (a\ b)\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}u$$

So we have a directional derivative in the direction (a,b). So u is a solution if a derivative of u in this direction is 0, i.e. u is constant along these lines, called the characteristic lines of the equation.



What is the equation of these lines?
They have slope $\frac{b}{a}$, i.e. $\frac{dy}{dx} = \frac{b}{a}$
which means that $y = \frac{b}{a}x + c$:

$$ay = bx + ca$$

$$\Rightarrow \quad bx - ay = \text{const}$$

So a solution u is constant along these lines:

$$u(x,y) = f(bx - ay)$$

Example: If $b=0$, w.l.o.g. $a=1$ so we have the PDE $u_x = 0$, i.e. u doesn't change in the x direction, i.e. u is only a function of y:

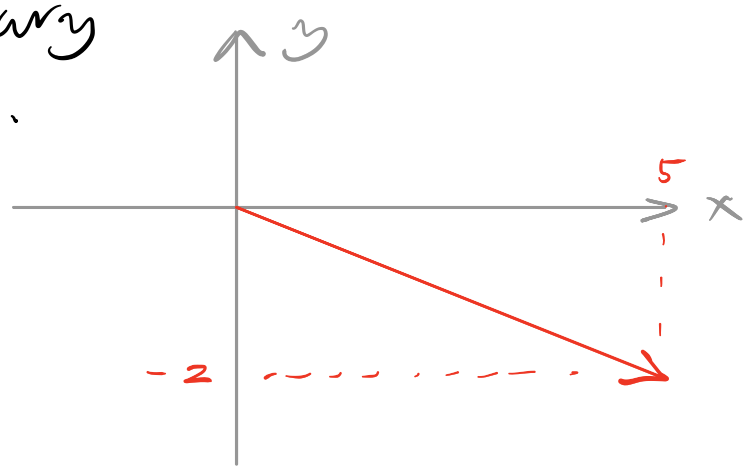
$$u(x,y) = f(y)$$

Can be anything: can be y, y^3, $y^9 + 50$
$\tan y$, $e^y + 3y + 7$.....

If we have the auxiliary condition $u(0,y) = e^y$ then we find that also $u(x,y) = e^y$ is the solution.

Example: let $a = 5$, $b = -2$

and consider the auxiliary condition $u(x,0) = \cos x$.



$$5u_x - 2u_y = 0$$

$$u(x,y) = f(-2x - 5y)$$

is the general solution.

$$u(x,0) = \cos x = f(-2x)$$

Substitute $w = -2x \implies f(w) = \cos\left(-\frac{w}{2}\right)$

Hence the solution is: $u(x,y) = \cos\left(x + \frac{5}{2}y\right)$

We can check:

$$5u_x - 2u_y = -5\sin\left(x+\tfrac{5}{2}y\right) + \cancel{2}\sin\left(x+\tfrac{5}{2}y\right)\cdot\frac{5}{\cancel{2}} = 0.$$

Variable coefficient case:

We can also have more general eq. of the form

$$a(x,y)u_x + b(x,y)u_y = 0.$$

These can be much more difficult to solve in general.

Strategy: this comes from the directional derivative

$$a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

The characteristic curves are given by the formula

$$-b(x,y)\,dx + a(x,y)\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

If we can solve this by integrating, we find $y = y(x)$, so that $(x, y(x))$ is the characteristic curve and u is constant along these curves: $u(x, y(x)) = \text{const.}$

Example: Consider $u_x + \cos x \, u_y = 0$.

We need to solve: $\frac{dy}{dx} = \frac{\cos x}{1}$

$$\Longrightarrow \quad y = \int \cos x \, dx = \sin x + c$$

Each value c will correspond to a different curve, so we have

$$u(x,y) = f(c) = f(y - \sin x)$$

where f is any function.

Check:
$$u_x = f'(y - \sin x) \cdot \frac{\partial}{\partial x}(y - \sin x)$$
$$= f'(y - \sin x)(-\cos x)$$
$$u_y = f'(y - \sin x) \cdot \frac{\partial}{\partial y}(y - \sin x)$$
$$= f'(y - \sin x) \cdot 1$$

$$\Longrightarrow \quad u_x + \cos x \, u_y = -f'(y - \sin x)\cos x + \cos x \, f'(y - \sin x)$$
$$= 0$$

If we are given the auxiliary cond. $u(0,y) = y^2$ then we have $y^2 = u(0,y) = f(y)$. So $f(c) = c^2$,

and $u(x,y) = (y - \sin x)^2$