MAGIC058: Theory of Partial Differential Equations

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These are the lecture notes of a graduate-level PDE course taught during the autumn (fall) semester of the 2023-24 academic year as part of the MAGIC consortium. The introduction is based on existing notes that have been circulating at Cardiff University, while I fashioned the subsequent chapters based on the books of F. John and W. A. Strauss. The intention was to create a graduate-level course that covers the three basic equations (wave, heat and Laplace) catering to students who may not have studied PDEs previously at all. For this reason there are two chapters covering Fourier series and separation of variables.

Jonathan Ben-Artzi

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Chapter 1

Introduction

1.1 Definitions

What is a PDE?

A **partial differential equation** (PDE) is an equation involving one or more partial derivatives of an unknown function of several variables. The PDE's **order** is the order of the highest-order derivative appearing in the equation.

The variables $x, y, \ldots$ denote independent *spatial variables*, whereas t denotes the *time variable* (if there is one). The dependent variables are typically denoted by $u(x, y, \ldots, t)$, $v(x, y, \ldots, t)$ and $w(x, y, \ldots, t)$. The explicit functional dependence will be sometimes omitted for simplicity. The following notation will be used interchangeably to denote partial derivatives

$$\frac{\partial u}{\partial x} \equiv u_x, \quad \frac{\partial u}{\partial y} \equiv u_y, \quad \frac{\partial^2 u}{\partial x^2} \equiv u_{xx}, \quad \frac{\partial^2 u}{\partial y^2} \equiv u_{yy}, \quad \frac{\partial^2 u}{\partial x \partial y} \equiv u_{xy}, \quad \ldots$$

as well as the usual symbol for the laplacian:

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \cdots$$

A PDE for a function $u(x, y, \ldots, t)$ has the general form

$$F(x, y, \ldots, t; u, u_x, u_y, , \ldots, u_t, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0.$$

The PDE is said to be:

Linear if F is algebraically linear in u and its partial derivatives and if the coefficients of u and its partial derivatives are functions of the independent variables only.

Semilinear if it is algebraically linear with respect to the highest derivatives and the coefficients of the highest derivatives depend on the independent variables only.

Quasilinear if it is algebraically linear with respect to the highest derivatives, but their coefficients may depend on lower derivatives as well as independent variables.

Nonlinear if it is none of the above.

Exercise 1.1: For each of the following PDEs state (i) the order of the PDE (ii) whether it is linear, semilinear, quasilinear or nonlinear.

(a) $u_t + uu_x = 0$ (b) $(a^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (a^2 - \phi_y^2)\phi_{yy} = 0$

(c) $u_x^2 + u_y^2 = 1$ (d) $u_{xx} = xu_{yy}$ (e) $u_t + uu_x = u_{xx}$

The independent variables, denoted $\mathbf{x} = (x, y, \ldots)$ for brevity, belong to the *domain* of the PDE, defined as follows:

Domain of a PDE

Steady-state problems

The **domain**, Ω, of a PDE of n spatial variables is a non-empty, open subset of $\mathbb{R}^n$. The domain Ω can be bounded or unbounded.

The **boundary** of Ω is denoted by $\partial\Omega$.

The **closure** of Ω, is denoted by $\bar{\Omega}$ and is the closed set defined as $\bar{\Omega} = \Omega \cup \partial\Omega$.

Time-dependent problems

If, in addition, a time variable is present, the domain of the PDE is an open subset of $\mathbb{R}^{n+1}$ defined as:

$$G = \Omega \times (0, +\infty) = \{(\mathbf{x}, t) : \mathbf{x} \in \Omega \quad \text{and} \quad t > 0\}.$$

A PDE defined on a bounded time interval is also possible, albeit less common.

1.2 Vector space structure

Operators

Given a PDE $F(x, y, \ldots; u, u_x, u_y, \ldots) = 0$, the associated **operator** is the mapping $\mathcal{L}$ that maps u to the part of F involving u.

An operator $\mathcal{L}$ is said to be **linear** if $\mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v)$ and $\mathcal{L}(au) = a\mathcal{L}(u)$ for any functions u, v, and for any constant a.

Let $\mathcal{L}$ be a linear operator. The equation

$$\mathcal{L}u = 0$$

is called a **homogeneous** linear equation. An equation of the form

$$\mathcal{L}u = g$$

where $g \neq 0$ is called an **inhomogeneous** linear equation.

Exercise 1.2: For each of the following PDEs find (i) the associated operator (ii) whether it is linear or nonlinear and (iii) whether it is homogeneous or inhomogeneous.

(a) $u_{tt} = c^2 u_{xx}$ (b) $u_{tt} = c^2 u_{xx} + f(x, t)$

(c) $u_x^2 + u_y^2 = 1$ (d) $u_{xx} = x u_{yy}$ (e) $u_t + u u_x = u_{xx}$

We observe that solutions admit a **vector space structure**: for a linear operator $\mathcal{L}$, if the functions $\{u_i\}_{i=1}^n$ solve the equation $\mathcal{L}u = 0$, then also $\mathcal{L}(\sum_{i=1}^n a_i u_i) = 0$ for any constants a_i, $i = 1, \dots, n$. Moreover, if v solves $\mathcal{L}v = g$ and u solves $\mathcal{L}u = 0$, then $u + v$ solves $\mathcal{L}(u + v) = g$. Hence, by solving the homogeneous equation we can find infinitely more solutions to an inhomogeneous equation.

1.3 Classification of second-order linear PDEs in two independent variables

Consider the general second-order linear PDE in the dependent variable u and the independent variables x and y,

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g, \tag{1.1}$$

where a, b, c, d, e, f and g are functions of x and y only which belong to $C^2(\Omega)$, with $\Omega \subset \mathbb{R}^2$. We also require that a, b and c do not vanish simultaneously at any point in Ω.

Remark: Here, the variables 'x' and 'y' don't necessarily have to be spatial variables. For instance, one of them could be a time variable.

The classification of PDEs is suggested by the corresponding classification of the quadratic equation of conic sections in analytic geometry. For example, the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

represents a hyperbola, parabola or ellipse, depending on whether the discriminant $D = b^2 - 4ac$ is positive, zero or negative, respectively. Likewise, if we consider the discriminant of (1.1) at some point $(x_0, y_0) \in \Omega$,

$$D(x_0, y_0) = b^2(x_0, y_0) - 4a(x_0, y_0)c(x_0, y_0),$$

then the PDE is said to be

- **hyperbolic** at (x_0, y_0), if $D(x_0, y_0) > 0$,
- **parabolic** at (x_0, y_0), if $D(x_0, y_0) = 0$,
- **elliptic** at (x_0, y_0), if $D(x_0, y_0) < 0$.

If any of the above is true for every point in Ω, then the equation is said to be hyperbolic, parabolic, or elliptic in a domain (Ω). In the same manner, we can also classify all semilinear PDEs of the form $a u_{xx} + b u_{xy} + c u_{yy} + F(x, y; u, u_x, u_y) = 0$, where F is arbitrary.

Exercise 1.3: Determine the type of the following PDEs

(a) $3u_{xx} + 2u_{xy} + 5u_{yy} + x u_y = 0$

(b) $u_{xx} = x u_{yy}$

(c) $3u_{xx} + u_{yy} + 4u_{zz} + 4u_{zy} = u_x$

1.4 Auxiliary Conditions

The PDEs that model physical systems typically have infinitely many solutions. To find the single function that is **the solution** to the physical problem, one needs to impose a set of auxiliary conditions. These conditions can be of two types:

Auxiliary conditions

Boundary Conditions (BCs) These are conditions that hold at points on $\partial\Omega$. Commonly the following types of conditions are used

Dirichlet condition $u = f$

Neumann (or flux) condition $\dfrac{\partial u}{\partial n} = g$

Robin (or mixed or radiation) condition $\alpha u + \beta \dfrac{\partial u}{\partial n} = g$

for some given functions α, β, f and g defined on $\partial\Omega$.

Initial Conditions (ICs) These are conditions that are used in time-dependent problems and specify u and its time derivatives at some initial time, typically at $t = 0$. Such conditions are also called **Cauchy** conditions.

Problems that are time-independent are called **boundary value problems** (BVP); otherwise they are called **initial-boundary value problems** (IBVP).

1.5 Classical solution

Given the PDE and its domain, together with the appropriate auxiliary conditions, we would like to characterise its solution (if it exists). This gives rise to the notion of a *classical solution* defined as follows:

Classical Solution

Consider a PDE of order k in n independent variables in an open subset G of $\mathbb{R}^n$ together with auxiliary conditions applied along ∂G. The function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be a **classical solution** to the problem if

1. u satisfies the PDE and auxiliary conditions pointwise for every point in the region where the problem is defined.
2. $u \in C^k(G)$.
3. continuity 'extends' to the boundary of the domain ∂G. For example, for problem with Dirichlet conditions we need to have $u \in C^0(\overline{G})$; for a problem with Neumann conditions we need to have $u \in C^1(\overline{G})$.

From the above, it is clear that a classical solution has certain smoothness requirements. Quite frequently, however, solutions may not exist in this (classical) sense and one needs to relax the smoothness conditions to find a solution; thus **weak solutions** or their derivatives are allowed to exhibit discontinuities in some or all points in G.

1.6 Well-posedness

A problem in PDEs is not guaranteed to have a solution, or, if does have *a* solution it is not guaranteed to be unique. According to the classical definition of Hadamard, a problem described by a PDE in a given domain together with a set of auxiliary conditions is said to be **well-posed** if the solution

1. exists,
2. is unique,
3. is stable (i.e. it depends continuously on the data[1]).

More specifically, note the following:

existence of the solution, means that it is possible to find at least one solution. This is typically harder to to establish theoretically. The most satisfactory way to show that a solution exists is to construct it.

uniqueness of the solution, means that with the given conditions we are able to find *at most* one solution.

stability of the solution, means that small perturbations in the data results in small perturbations in the solution.

If *any* of the above conditions is not satisfied, then the problem is said to be **ill-posed**.

EXAMPLE 1.1: The following problem is due to Hadamard: Consider the initial-value problem for the Laplace equation

$$\Delta u = 0, \quad 0 < y < \infty, \quad x \in \mathbb{R}.$$

with the Cauchy data

$$u(x,0) = 0 \quad \text{and} \quad u_y(x,0) = \frac{\sin mx}{m},$$

where m is an integer representing the wavenumber. These data tend to zero uniformly as $m \to \infty$. It can be easily verified (more about this later in the module) that

$$u(x,y) = \frac{\sinh my \sin mx}{m^2}$$

is the *unique* solution to the problem. Is this problem well-posed?

[1]The data of the problem corresponds to the inhomogeneous term in the PDE together with the prescribed functions in the initial and boundary conditions.

SOLUTION
The problem
$$\Delta u = 0, \quad 0 < y < \infty, \quad x \in \mathbb{R}.$$
with the Cauchy data
$$u(x,0) = 0 \quad \text{and} \quad u_y(x,0) = 0,$$
has a unique solution $u(x,y) \equiv 0$. The solution to the original problem represents oscillations in x with unbounded amplitude $m^{-2}\sinh my$, which tends to infinity as $m \to \infty$. In other words, even though the data tends to zero uniformly as $m \to \infty$, the solution itself does not tend to the solution $u(x,y) \equiv 0$, as it should.

In this sense, the solution is unstable since $u(x,y) \to \infty$ as $m \to \infty$ for any fixed point (x,y) such that $y > 0$. Hence the problem is ill-posed. ■

Exercise 1.4: Consider the following problems:

(a) $u_{xx} = 0$ for $0 < x < 1$ with BCs $u_x(1) = 1$ and $u_x(0) = 0$.

(b) $u_{xy} = 0$ in $\Omega = (0,\infty) \times (0,\infty)$ with BCs $u(x,0) = x$ and $u(0,y) = y$.

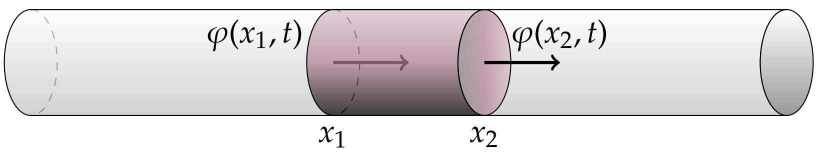
(c) $\Delta u = 0$ in some open $\Omega \subset \mathbb{R}^3$ with $\partial u/\partial n = c$ along $\partial\Omega$, where c is a constant. (**Hint**: use the divergence theorem)

In each case, discuss the well-posedness of the problem in terms of the existence, uniqueness and stability of their solution(s) (if they exist). When discussing stability, only consider perturbations to the BC; do not add an inhomogeneous term in the PDE.

1.7 Some basic linear PDEs and their applications

1.7.1 The heat equation (also known as the *diffusion* equation)

Consider the conduction of heat in a one-dimensional rod (a thin cylinder with its cylindrical surface insulated). Let L be the length of the rod, A the cross sectional area of the rod (assumed constant) and $u(x,t)$ the temperature of the rod at location x and time t.



Focusing on a short segment of the rod located between x_1 and x_2, we have that the total internal energy within the rod is given by
$$\int_{x_1}^{x_2} E(x,t)\ \mathrm{d}x,$$
where $E(x,t)$ corresponds to the heat energy per unit length. It is related to the temperature through the linear relation
$$E(x,t) = c(x)\rho(x)u(x,t)A$$
which reflects the fact that the change in the internal energy corresponds to the change in temperature (here $u(x,t)$ is measured in relation to the absolute zero temperature). Here $\rho(x)$ is the mass density of the rod and $c(x)$ is the specific heat which corresponds to the energy required to raise the temperature of one unit of mass by one unit.

To set up the model, we invoke the conservation of energy which says that

$$\left\{\begin{array}{c}\text{Change of}\\ \text{internal energy}\\ \text{in time } \delta t\end{array}\right\} = \left\{\begin{array}{c}\text{heat flow}\\ \text{across the body}\\ \text{in time } \delta t\end{array}\right\} + \left\{\begin{array}{c}\text{heat produced/removed}\\ \text{in time } \delta t\\ \text{by external factors}\end{array}\right\}$$

This is expressed mathematically as

$$\begin{aligned}\int_{x_1}^{x_2} E(x,t+\delta t)\ \mathrm{d}x - \int_{x_1}^{x_2} E(x,t)\ \mathrm{d}x &= [\varphi(x_1,t)-\varphi(x_2,t)]A\delta t + \delta t \int_{x_1}^{x_2} q(x,t)A\ \mathrm{d}x\\ &= -A\delta t \int_{x_1}^{x_2} \varphi_x(x,t)\ \mathrm{d}x + A\delta t \int_{x_1}^{x_2} q(x,t)\ \mathrm{d}x,\end{aligned}$$

where $\varphi(x,t)$ is the heat flux (heat energy per unit area) and $q(x,t)$ represents the external heat sources or sinks. After some term re-arrangement we find

$$\int_{x_1}^{x_2} \left[c(x)\rho(x)\frac{u(x,t+\delta t)-u(x,t)}{\delta t} + \varphi_x(x,t) - q(x,t)\right]\ \mathrm{d}x = 0$$

Taking the limit $\delta t \to 0$ and the arbitrariness of x_1 and x_2 yield

$$c(x)\rho(x)u_t(x,t) + \varphi_x(x,t) = q(x,t) \quad \text{for} \quad 0 < x < L \quad \text{and} \quad t > 0. \tag{1.2}$$

To continue, we invoke the so-called Fourier's law of heat conduction which relates the heat flux with the temperature gradient

$$\varphi(x,t) = -K(x)u_x(x,t),$$

where $K(x)$ is the thermal conductivity, so that (1.2) becomes

$$c(x)\rho(x)u_t(x,t) = (K(x)u_x(x,t))_x + q(x,t) \quad \text{for} \quad 0 < x < L \quad \text{and} \quad t > 0. \tag{1.3}$$

If $c(x)$, $\rho(x)$ and $K(x)$ are assumed to be constants and there are no heat sources (i.e. $q(x,t) \equiv 0$), (1.3) becomes

$$u_t = \kappa u_{xx} \quad \text{for} \quad 0 < x < L \quad \text{and} \quad t > 0, \tag{1.4}$$

where $\kappa = K/(c\rho)$ is the thermal diffusivity of the rod. Equation (1.4) is called the **heat equation**.

Remarks

- The heat equation (1.4) is a prototypical example of a *parabolic* PDE.
- This equation is first-order in time and second-order in space. To solve it, we need to impose one initial condition, i.e. $u(x,0)$ which corresponds to the initial temperature distribution and two boundary conditions.
- For the boundary conditions there can be three physically meaningful boundary conditions at an endpoint $x = a$, where a is either 0 or L:

1. The Dirichlet condition

$$u(a,t) = g(t), \quad t > 0$$

fixes the temperature at $x = a$.

2. The Neumann condition

$$u_x(a,t) = f(t), \quad t > 0$$

prescribes the heat flux through $x = a$. If $f(t) \equiv 0$ means that the rod is *insulated* at $x = a$ (no heat flux).

3. The Robin condition

$$\pm K_0 u_x(a,t) = H\left[u(a,t) - U(t)\right], \quad t > 0$$

is used to model the heat transfer occurring by bringing the endpoint in contact with another medium, where $U(t)$ is the (known) temperature of the medium and $H > 0$ is the heat transfer coefficient. Due to the convention used for the heat flux, we use the '+' sign when $a = 0$ and the '−' sign when $a = L$.

4. Only one boundary condition can be prescribed at each endpoint; the conditions need not be identical (e.g. one end might be insulated whereas the other kept at constant temperature).

1.7.2 Laplace's equation

The considerations in the previous section may be generalised to higher dimensions. For simplicity we will only look at the two-dimensional case. In this case (1.3) may be generalised to give the temperature distribution $u(x,y,t)$ of a heat-conducting body occupying a region Ω of the (x,y)-plane:

$$c(x,y)\rho(x,y)u_t(x,y,t) = \nabla \cdot \left[K(x,y)\nabla u(x,y,t)\right] + q(x,y,t) \quad \text{in} \quad \Omega. \tag{1.5}$$

If $c = \rho \equiv 1$ and K is assumed to be constant in Ω (1.5) becomes

$$u_t(x,y,t) = K\,\Delta\,u(x,y,t) + q(x,y,t) \quad \text{in} \quad \Omega. \tag{1.6}$$

Equation (1.6) is the two-dimensional heat equation with external heat sources (or sinks).

If $q(x,y,t)$ and the boundary conditions attain a steady-state as $t \to \infty$, (1.6) reaches an *equilibrium temperature*, which satisfies an equation of the form

$$\Delta\,u(x,y) = g(x,y) \quad \text{in} \quad \Omega, \tag{1.7}$$

where we set $g(x,y) = -K^{-1}\lim_{t\to\infty} q(x,y,t)$. Equation (1.7) is called the **Poisson equation**. When we take $q(x,y) \equiv 0$, the PDE

$$\Delta\,u(x,y) = 0 \quad \text{in} \quad \Omega \tag{1.8}$$

is called the **Laplace equation**.

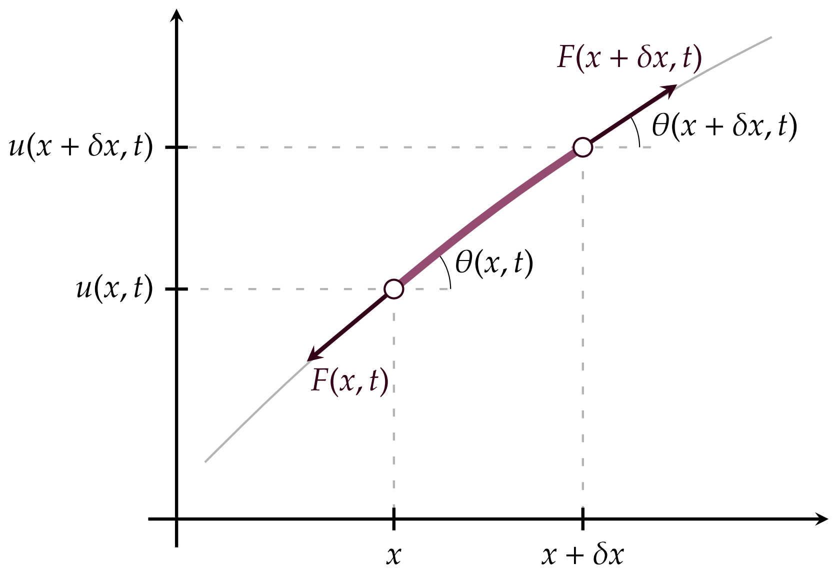
Remarks

- The Laplace equation is a prototypical example of an *elliptic* PDE.
- The Laplace and Poisson equations arise in different contexts. In electrostatics, the solution to the Poisson equation (1.7) gives the electrostatic potential $u(x, y)$ with $g(x, y) = -\rho(x, y)/\varepsilon_0$, where $\rho(x, y)$ is the prescribed charge density and ε_0 is the vacuum permitivity; in non-viscous fluid dynamics the Laplace equation $\Delta\phi = 0$ gives the velocity potential $\phi(x, y)$, from which one can deduce the velocity of the fluid on the (x, y)-plane as

$$\mathbf{v}(x, y) = \phi_x(x, y)\mathbf{i} + \phi_y(x, y)\mathbf{j}.$$

1.7.3 The wave equation

Consider the motion of a tightly stretched elastic string of length L, mass density (mass per unit length) $\rho(x)$ under the influence of a vertical body force $q(x, t)$ per unit mass. We would like to develop a model for the vertical displacement of the string, $u(x, t)$, under the assumption that the horizontal displacement of the string is negligible.



Focusing on a small segment of the string between points x and $x + \delta x$, consider Newton's second law along the vertical direction

$$\{\text{force}\} = \{\text{mass}\} \times \{\text{acceleration}\}$$

so that we have approximately

$$\underbrace{\rho(x)\,\delta x}_{\text{mass}}\, u_{tt}(x, t) \approx F(x + \delta x, t) \sin\theta(x + \delta x, t) - F(x, t) \sin\theta(x, t) + \rho(x)\,\delta x\, q(x, t).$$

If we divide through by δx and let $\delta x \to 0$ we find

$$\rho(x)u_{tt}(x, t) = [F(x, t) \sin\theta(x, t)]_x + \rho(x)q(x, t),$$

For small, θ we can use the approximation

$$u_x = \tan\theta \approx \theta \approx \sin\theta,$$

so that

$$\rho(x)u_{tt}(x,t) = [F(x,t)u_x(x,t)]_x + \rho(x)q(x,t), \tag{1.9}$$

For homogeneous (i.e. $\rho(x) = \rho_0 = \text{const.}$) and perfectly elastic strings (i.e. we may take $F(x,t) = F_0 = \text{const.}$), (1.9) becomes

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) + q(x,t), \tag{1.10}$$

where $c^2 = F_0/\rho_0$ has dimensions of velocity. If, the body force is assumed to be negligible compared to the string tension so that we take $q(x,t) \equiv 0$, (1.10) becomes

$$u_{tt}(x,t) = c^2 u_{xx}(x,t). \tag{1.11}$$

Equation (1.11) is called the one-dimensional **wave equation** and the constant c is the *wave speed.*

Remarks

- The wave equation is a prototypical example of a *hyperbolic* PDE
- This equation is second-order in time and second-order in space. To solve it, we need to impose two initial conditions (the initial position and velocity of points in the string, i.e. $u(x,0)$ and $u_t(x,0)$, respectively) and two boundary conditions.
- As in the case of the heat equation, there are three physically meaningful possible boundary conditions at an endpoint $x = a$, where a is either 0 or L:

  1. The Dirichlet condition

     $$u(a,t) = g(t), \quad t > 0$$

     prescribes the displacement at $x = a$.
  2. The Neumann condition

     $$u_x(a,t) = 0, \quad t > 0$$

     is used when the endpoint is free (no vertical tension).
  3. The Robin condition

     $$\pm F_0 u_x(a,t) = ku(a,t), \quad t > 0$$

     is used to model the case when the endpoint has an elastic attachment. We use the '+' sign when $a = 0$ and the '−' sign when $a = L$.
  4. Only one boundary condition can be prescribed at each endpoint; the conditions need not be identical (e.g. one end might be free to move whereas the other may be kept fixed at all times).

1.7.4 Other linear equations

Brownian motion / Convection-Diffusion problems

The PDE for $u(x, y, t)$

$$u_t = a\,\Delta\, u - \mathbf{v} \cdot \nabla u, \tag{1.12}$$

where a is constant corresponding to the diffusion coefficient and $\mathbf{v}$ is a constant vector that measures the drift velocity. This PDE may be used in different contexts:

Brownian motion: to describe the probability density function of the velocity of a particle at a point (x, y) at at time t, as it undergoes two-dimensional random motion in a fluid. In this context, (1.12) is a simplified version of the *Fokker–Planck* equation.

Convection-diffusion: to describe physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to diffusion and convection. For example, in heat transfer $u(x, y, t)$ in (1.12) is used to denote the temperature field; in mass transfer $u(x, y, t)$ may denote, for example, the species concentration.

It is easy to verify that (1.12) is a parabolic PDE; for the steady-state problem ($u_t \equiv 0$), (1.12) is an elliptic PDE.

Stock market prices

The price $V(s, t)$ of a derivative on the stock market[2] depends on the stock price s and time t in years and is found by solving the so-called **Black-Scholes equation**

$$V_t + \frac{1}{2}\sigma^2 s^2 V_{ss} + rsV_s - rV = 0$$

where σ is the standard deviation of the stock returns and r is the annualised risk-free interest rate.

Quantum mechanics

In quantum mechanics, linear PDEs are used to describe the quantum state of a moving particle by the determining the associated wavefunction $\psi(x, y, t)$.

Non-relativistic quantum mechanics: in this case, the wavefunction satisfies the parabolic *Schrödinger equation*

$$\mathrm{i}\hbar\psi_t(x, y, t) = -\frac{\hbar^2}{2m}\,\Delta\,\psi(x, y, t) + V(x, y, t)\psi(x, y, t),$$

where $V(x, y, t)$ is an applied potential, m is the mass of the particle, $\hbar$ is the reduced Planck constant and $\mathrm{i}^2 = -1$.

[2] Derivatives in finance correspond to contracts whose value is determined/derived by other financial instruments.

Relativistic quantum mechanics: here the wavefunction satisfies the (hyperbolic) *Klein-Gordon equation*

$$\psi_{tt}(x, y, t) = c^2 \Delta \psi(x, y, t) - \frac{m^2 c^4}{\hbar^2} \psi(x, y, t),$$

where c is the speed of light.

Wave propagation

The second-order hyperbolic PDE of the from

$$u_{tt}(x,t) + a u_t(x,t) + b u(x,t) = c^2 u_{xx}(x,t) - d u_x(x,t), \tag{1.13}$$

where $a, b, d \geq 0$ are not all zero and $c > 0$ are used in different settings to study wave propagation. For example

Lossy transmission lines: The current and voltage in the propagation of signals is studied with the help of (1.13) in the case when $d = 0$. The remaining constants a, b and c are expressed in terms of the resistance, capacitance, conductance and inductance characterising the transmission line. This equation is called the *telegrapher's equation.*

Dissipative wave dynamics: The more general equation (1.13) is used to study waves that dissipate as they propagate. Note that we recover the (non-dissipative) wave equation, (1.11), if we set $a = b = d = 0$.

Wave scattering

The *Helmholtz equation*

$$\Delta u(x, y, z) + k^2 u(x, y, z) = 0$$

where $k =$ const. plays an important role in the study of scatttering of acoustic, electromagnetic and elastic waves. It is a second-order elliptic PDE and arises by considering the three-dimensional wave equation

$$v_{tt}(x, y, z, t) = \Delta v(x, y, z, t),$$

and seeking solutions of the form

$$v(x, y, z, t) = u(x, y, z) \mathrm{e}^{\mathrm{i}kt}.$$

Transverse vibrations of a rod

The deflection of $u(x, t)$ of a point x in the rod at time t satisfies the fourth-order PDE:

$$u_{tt}(x,t) + c^2 u_{xxxx}(x,t) = 0.$$

where c is a constant associated with the rigidity of the rod.

The two-dimensional biharmonic equation

The biharmonic equation

$$\Delta^2 \psi(x,y) = \psi_{xxxx}(x,y) + 2\psi_{xxyy}(x,y) + \psi_{yyyy}(x,y) = 0$$

(also written $\nabla^4 \psi = 0$ sometimes) is a fundamental equation in continuum mechanics. In plane elasticity, $\psi(x,y)$ denotes the so-called Airy stress function; in the slow viscous flow of an incompressible fluid (also called Stokes flow) $\psi(x,y)$ corresponds to the stream function of the flow, with the velocity vector given by

$$\mathbf{v}(x,y) = \psi_y(x,y)\mathbf{i} - \psi_x(x,y)\mathbf{j}.$$

Plane transonic flow

The transonic flow of a compressible gas is described by the Euler-Tricomi equation

$$u_{xx}(x,y) = x u_{yy}(x,y)$$

where $u(x,y)$ is the speed of the flow.

1.8 Review

Can you do the following?

- Distinguish between
  - Linear, semilinear, quasilinear and nonlinear PDEs;
  - Parabolic, Elliptic and Hyperbolic PDEs;
  - Boundary (Dirichlet, Neumann and Robin) and initial conditions;
  - Classical and weak solutions;
- Classify second-order PDEs using the discriminant method;
- Obtain general solutions to simple PDEs;
- Define what it means for a problem to be well-posed;
- Identify the three fundamental second-order PDEs (heat, Laplace and wave equations) and the physical interpretation of the Dirichlet and Neumann boundary conditions.

Chapter 2

The Heat (Diffusion) Equation on $\mathbb{R}$

The heat equation (also known as the *diffusion* equation) in one spatial dimension is given by

$$u_t(x,t) - \kappa u_{xx}(x,t) = 0 \quad \text{for} \quad -\infty < x < +\infty \quad \text{and} \quad t > 0, \tag{2.1}$$

where $\kappa > 0$ is a constant capturing physical properties of the problem at hand (see (1.4)). This equation can be written with $d \in \mathbb{N}$ spatial variables as well, where ∂_{xx} is replaced by the Laplacian Δ and the equation becomes

$$u_t(\mathbf{x},t) - \kappa \Delta u(\mathbf{x},t) = 0 \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^d \quad \text{and} \quad t > 0,$$

however here we concentrate on the one-dimensional case for simplicity, as it already exhibits all the interesting aspects of the heat equation.

The goal of this section is to obtain an explicit expression for the solution of the inhomogeneous Dirichlet problem

$$\begin{cases} u_t - \kappa u_{xx} = f & \text{in} \quad (x,t) \in \mathbb{R} \times (0,+\infty), \\ \qquad\quad u = g & \text{on} \quad (x,t) \in \mathbb{R} \times \{t = 0\}, \end{cases}$$

where f and g must satisfy some regularity conditions. The explicit expression we obtain (see (2.5) below) shows that there exists a solution. Moreover, from the explicit expression it is not hard to also see that it is stable: 'small' changes in f and g induce 'small' changes in the solution. It is possible to also show that the solution is unique among 'nice'[1] solutions (there exist other, 'bad' solutions). Hence (within the class of 'good' solutions) this is a **well-posed** problem.

[1]'Nice' solutions are solutions that do not grow too fast as $|x| \to +\infty$.

2.1 The Fundamental Solution

The Fundamental Solution

The function

$$\Phi(x,t) = \begin{cases} \frac{1}{2\sqrt{\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}} & -\infty < x < +\infty \quad \text{and} \quad t > 0, \\ 0 & -\infty < x < +\infty \quad \text{and} \quad t < 0, \end{cases}$$

is called the **fundamental solution** of the heat equation. It is also known as the **heat kernel**. It is an example of a *positive summability kernel*.

Lemma 2.1: The fundamental solution Φ satisfies

$$\int_{-\infty}^{\infty} \Phi(x,t)\ \mathrm{d}x = 1 \quad \text{for all} \quad t > 0,$$

and for any $\delta > 0$

$$\int_{|x|>\delta} \Phi(x,t)\ \mathrm{d}x \to 0 \quad \text{as} \quad t \downarrow 0.$$

Proof. This is a simple calculation:

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(x,t)\ \mathrm{d}x &= \frac{1}{2\sqrt{\pi\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\kappa t}}\ \mathrm{d}x \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}\ \mathrm{d}q \\ &= 1. \end{aligned}$$

The second part is left as an exercise. □

Lemma 2.2: For any $t > 0$, the fundamental solution Φ is a solution of (2.1).

Proof. The proof is left as a simple exercise. □

Exercise 2.1: (a) Prove that $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}\ \mathrm{d}q = 1$ (**Hint**: square it).

(b) Prove the second part of Lemma 2.1.

(c) Prove Lemma 2.2.

The fundamental solution Φ is therefore a function that always has area 1 under its graph, for any time $t > 0$. In Figure 2.1 we see the general behavior of Φ: for small times it is concentrated around $x = 0$, whereas for large times it spreads.

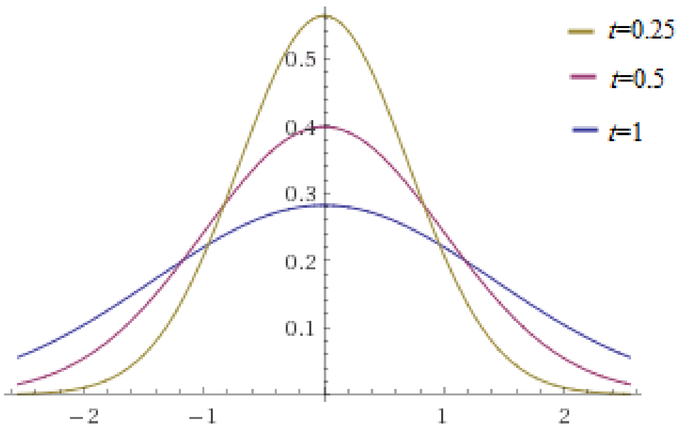


Figure 2.1: The function $\Phi(x, t)$ at different times. Notice that as $t \downarrow 0$, $\Phi(\cdot, t)$ becomes more concentrated near $x = 0$.

Approximation of the Dirac δ distribution

The δ distribution (due to physicist Paul Dirac) is often thought of as a 'function' satisfying $\delta(x) = 0$ for all $x \neq 0$ and $\delta(0) = +\infty$. Its usefulness is when integrating it against continuous functions: $\int_{-\infty}^{\infty} \delta(x) f(x) \, \mathrm{d}x = f(0)$. It is easier to understand this object via smooth functions that approximate it. These are called *positive summability kernels*, of which the heat kernel is a prime example.

The heat kernel Φ approximates the δ distribution:

$$\Phi(\cdot, t) \to \delta \quad \text{in the sense of distributions as} \quad t \downarrow 0.$$

This is a rigorous way to express what is evident from Figure 2.1: as $t \downarrow 0$, $\Phi(\cdot, t)$ becomes ever more concentrated around 0, while always maintaining area 1 under its graph.

Proof. The theory of *distributions* is beyond the scope of this course. This is left as an exercise (essentially an exercise in integration by parts). Alternatively, see F. John, *Partial Differential Equations, Fourth Edition*, Chapter 3.6 or W. A. Strauss, *Partial Differential Equations: An Introduction, Second Edition*, Chapter 12.1. □

Brownian Motion

The heat kernel has a critical physical interpretation. Taking $\kappa = \frac{1}{2}$, the fundamental solution for $t > 0$ beomes $\Phi(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ which is precisely the probability density function corresponding to a one-dimensional brownian motion which starts at $x = 0$ at time $t = 0$. See W. A. Strauss, *Partial Differential Equations: An Introduction, Second Edition*, Chapter 2.4 for further discussion.

2.2 The Initial-Value Problem

We now wish to solve the *initial-value (Cauchy) problem*:

$$\begin{cases} u_t - \kappa u_{xx} = 0 & \text{in} \quad (x,t) \in \mathbb{R} \times (0,+\infty), \\ u = g & \text{on} \quad (x,t) \in \mathbb{R} \times \{t = 0\}. \end{cases}$$

As the next theorem shows, the fundamental solution is the key: it gives us a solution for all $t > 0$ in terms of the initial condition g. The tricky part is to show that as $t \downarrow 0$, u indeed converges to g.

Theorem 2.3: Let $g \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then the function $u : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ defined as

$$u(x,t) := \Phi(\cdot,t) *_x g(\cdot) = \int_{-\infty}^{\infty} \Phi(x-y,t) g(y) \, \mathrm{d}y \tag{2.2}$$

is a C^∞ solution of (2.1) and satisfies

$$\lim_{t\downarrow 0} u(\xi,t) = g(\xi) \quad \text{for every} \quad \xi \in \mathbb{R}.$$

Proof. The function u is defined as the convolution of an infinitely-differentiable function and a bounded function, and is therefore infinitely-differentiable itself. Plugging the expression for u into the equation (2.1) we find

$$\begin{aligned} u_t - \kappa u_{xx} &= \int_{-\infty}^{\infty} [\Phi_t - \kappa \Phi_{xx}] (x-y,t) g(y) \, \mathrm{d}y \\ &= 0 \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0 \end{aligned}$$

due to Lemma 2.2.

It remains to show the limit as $t \downarrow 0$. Since for small $t > 0$ the function Φ is concentrated around 0, we shall break up the integral appearing in (2.2) into two parts: one near ξ and the other away from ξ. Fix $\xi \in \mathbb{R}$ and $\varepsilon > 0$. Choose $\delta > 0$ such that

$$|g(y) - g(\xi)| < \varepsilon \quad \text{if} \quad y \in \mathbb{R} \quad \text{is such that} \quad |y - \xi| < \delta.$$

Exercise 2.2: Why does there exist such a δ?

Then we compute the difference

$$\begin{aligned} |u(\xi,t) - g(\xi)| &= \left| \int_{-\infty}^{\infty} \Phi(\xi - y, t) g(y) \, \mathrm{d}y - g(\xi) \right| \\ &= \left| \int_{-\infty}^{\infty} \Phi(\xi - y, t)(g(y) - g(\xi)) \, \mathrm{d}y \right| \\ &\le \underbrace{\left| \int_{|\xi-y|<\delta} \Phi(\xi - y, t)(g(y) - g(\xi)) \, \mathrm{d}y \right|}_{I_1} + \underbrace{\left| \int_{|\xi-y|>\delta} \Phi(\xi - y, t)(g(y) - g(\xi)) \, \mathrm{d}y \right|}_{I_2}. \end{aligned}$$

Exercise 2.3: Can you explain the second equality?

From Lemma 2.1 we know that $I_2 \to 0$ as $t \downarrow 0$. From our choice of δ we know that $I_1 < \varepsilon$. Since this is true for any $\varepsilon > 0$, we conclude that $\lim_{t\downarrow 0} u(\xi,t) = g(\xi)$. □

Remark: Observe that the only place where the continuity of g was used was in showing that $\lim_{t\downarrow 0} u(\xi,t) = g(\xi)$. We could have removed the requirement that g is continuous, in which case the statement that $\lim_{t\downarrow 0} u(\xi,t) = g(\xi)$ would only hold at points of continuity of g. However it would still hold that the solution is C^∞ in $\mathbb{R}\times(0,+\infty)$, which is a remarkable fact: any discontinuities in the initial condition will immediately be smoothed and become infinitely differentiable!

2.3 The Inhomogeneous Problem

We now wish to solve the inhomogeneous problem

$$\begin{cases} u_t - \kappa u_{xx} = f & \text{in} \quad (x,t)\in\mathbb{R}\times(0,+\infty), \\ u = 0 & \text{on} \quad (x,t)\in\mathbb{R}\times\{t=0\}. \end{cases} \tag{2.3}$$

Duhamel's Principle

The solution to (2.3) can be built from f by taking a **convolution in space and time**. The resulting expression is

$$u(x,t) = \Phi *_{x,t} f = \int_0^t \int_{-\infty}^{\infty} \Phi(x-y,t-s)f(y,s)\ \mathrm{d}y\ \mathrm{d}s. \tag{2.4}$$

Theorem 2.4: Assume that f is a compactly supported function such that f, f_x, f_{xx}, f_t are all continuous in $\mathbb{R}\times[0,+\infty)$. Then u defined as in (2.4) is a classical solution:

(a) u, u_x, u_{xx}, u_t are all continuous in $\mathbb{R}\times(0,+\infty)$;

(b) u satisfies the equation $u_t - \kappa u_{xx} = f$ in $\mathbb{R}\times(0,+\infty)$;

(c) $\lim_{t\downarrow 0} u(\xi,t) = 0$ for every $\xi\in\mathbb{R}$.

Proof. Recall that Φ approximates the δ distribution; consequently, Φ is singular at $(0,0)$ and one must be cautious when trying to take derivatives of the expression for u. We circumvent this by recalling that the convolution is commutative, so that we can write

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \Phi(y,s)f(x-y,t-s)\ \mathrm{d}y\ \mathrm{d}s.$$

Recalling the definition of a derivative, we first write the finite difference

$$\frac{u(x+h,t)-u(x,t)}{h} = \int_0^t \int_{-\infty}^{\infty} \Phi(y,s)\frac{f(x+h-y,t-s)-f(x-y,t-s)}{h}\ \mathrm{d}y\ \mathrm{d}s.$$

Now, since f has compact support and both f and f_x are continuous in $\mathbb{R}\times[0,+\infty)$ we have that

$$\frac{f(x+h-y,t-s)-f(x-y,t-s)}{h} \to f_x(x-y,t-s)$$

uniformly in $\mathbb{R} \times [0, +\infty)$ as $h \to 0$. Consequently, we can take the limit inside the integral, and we obtain

$$u_x(x,t) = \int_0^t \int_{-\infty}^{\infty} \Phi(y,s) f_x(x-y, t-s)\ \mathrm{d}y\ \mathrm{d}s.$$

Similar reasoning leads to

$$u_{xx}(x,t) = \int_0^t \int_{-\infty}^{\infty} \Phi(y,s) f_{xx}(x-y, t-s)\ \mathrm{d}y\ \mathrm{d}s.$$

For the t derivative we also have to differentiate the outer integral, which leads to

$$\begin{aligned} u_t(x,t) = &\int_0^t \int_{-\infty}^{\infty} \Phi(y,s) f_t(x-y, t-s)\ \mathrm{d}y\ \mathrm{d}s \\ &+ \int_{-\infty}^{\infty} \Phi(y,t) f(x-y,0)\ \mathrm{d}y. \end{aligned}$$

This concludes the proof of (a). To prove part (b) we proceed by writing the expression $u_t - \kappa u_{xx}$ with the hope of finding that it equals f:

$$u_t - \kappa u_{xx} = \underbrace{\int_0^t \int_{-\infty}^{\infty} \Phi(y,s)(f_t - \kappa f_{xx})(x-y, t-s)\ \mathrm{d}y\ \mathrm{d}s}_{I} + \underbrace{\int_{-\infty}^{\infty} \Phi(y,t) f(x-y,0)\ \mathrm{d}y}_{K}.$$

We would like to integrate the term I by parts since we know that Φ satisfies the homogeneous heat equation; however the singularity of Φ at $(0,0)$ is a problem. So we split the time integral into a 'bad' part (near 0) and a 'good' part (away from 0):

$$\begin{aligned} I = &\int_0^{\varepsilon} \int_{-\infty}^{\infty} \Phi(y,s)(f_t - \kappa f_{xx})(x-y, t-s)\ \mathrm{d}y\ \mathrm{d}s \\ &+ \int_{\varepsilon}^{t} \int_{-\infty}^{\infty} \Phi(y,s)(f_t - \kappa f_{xx})(x-y, t-s)\ \mathrm{d}y\ \mathrm{d}s =: I_{Bad} + I_{Good}. \end{aligned}$$

The bad term is small:

$$\begin{aligned} |I_{Bad}| =& \left| \int_0^{\varepsilon} \int_{-\infty}^{\infty} \Phi(y,s)(f_t - \kappa f_{xx})(x-y, t-s)\ \mathrm{d}y\ \mathrm{d}s \right| \\ =& \left| \int_0^{\varepsilon} \int_{-\infty}^{\infty} \Phi(y,s)(-f_s - \kappa f_{yy})(x-y, t-s)\ \mathrm{d}y\ \mathrm{d}s \right| \\ \leq& \left(\|f_t\|_{L^\infty} + \kappa \|f_{xx}\|_{L^\infty} \right) \int_0^{\varepsilon} \int_{-\infty}^{\infty} \Phi(y,s)\ \mathrm{d}y\ \mathrm{d}s \\ \leq& C\varepsilon. \end{aligned}$$

The good term can be integrated by parts safely, and we can use the fact that Φ satisfies the homogeneous heat equation:

$$\begin{aligned}
I_{Good} &= \int_{\varepsilon}^{t}\int_{-\infty}^{\infty} \Phi(y,s)(-f_s - \kappa f_{yy})(x-y,t-s)\ dy\ ds \\
&= \int_{\varepsilon}^{t}\int_{-\infty}^{\infty} \left(\Phi_s - \kappa\Phi_{yy}\right)(y,s) f(x-y,t-s)\ dy\ ds \\
&\quad + \int_{-\infty}^{\infty} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\ dy \\
&\quad - \int_{-\infty}^{\infty} \Phi(y,t) f(x-y,0)\ dy \\
&= \int_{-\infty}^{\infty} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\ dy - K,
\end{aligned}$$

where we recall that K was defined above already. We finally conclude that

$$u_t - \kappa u_{xx} = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\ dy = f(x,t)$$

where the limit follows an argument similar to the one appearing in the proof of Theorem 2.3. Part (c) follows trivially directly from (2.4): $\|u(\cdot,t)\|_{L^\infty} \le t\|f\|_{L^\infty}$ which tends to 0 as $t \downarrow 0$. □

Exercise 2.4: Can you show the last limit above (where $\varepsilon \to 0$)?

2.4 The Inhomogeneous Initial-Value Problem

By the vector-space structure, it follows that the solution to

$$\begin{cases} u_t - \kappa u_{xx} = f & \text{in} \quad (x,t) \in \mathbb{R}\times(0,+\infty), \\ u = g & \text{on} \quad (x,t) \in \mathbb{R}\times\{t=0\}. \end{cases}$$

(where f and g satisfy the same hypotheses as above) is given by

$$u(x,t) = \Phi(\cdot,t) *_x g + \Phi *_{x,t} f. \tag{2.5}$$

2.5 Uniqueness of Solutions

The uniqueness of solutions of (2.1) is not as straightforward as may seem. It turns out that the explicit solution (2.5) is *not* unique. It is, however, the unique *physical* solution.

Proposition 2.5: The initial value problem

$$\begin{cases} u_t - \kappa u_{xx} = 0 & \text{in} \quad (x,t) \in \mathbb{R}\times(0,+\infty), \\ u = 0 & \text{on} \quad (x,t) \in \mathbb{R}\times\{t=0\}, \end{cases}$$

has infinitely many solutions that are infinitely differentiable in both x and t (for all t, including $t<0$) which vanish identically for $t<0$.

Proof. We first note that this problem has one obvious solution: the trivial solution $u \equiv 0$. That is also the *physical* solution. However, as we shall now see, there are *non-physical* solutions. The starting point is to obtain formal solutions, and then justify their validity rigorously. The proof is split into several steps.

1. A formal expression. For simplicity we take $\kappa = 1$. Write

$$u(x,t) = \sum_{j=0}^{\infty} h_j(t) x^j \tag{2.6}$$

for some functions h_j, to be determined shortly. Formally, we have

$$u_t = \sum_{j=0}^{\infty} h_j'(t) x^j \quad \text{and} \quad u_{xx} = \sum_{j=0}^{\infty} (j+2)(j+1) h_{j+2}(t) x^j$$

which leads to

$$h_0 = h, \quad h_1 = 0, \quad h_j'(t) = (j+2)(j+1)h_{j+2}(t) \quad \text{for} \quad j = 0, 1, \ldots$$

where h is some function. Plugging this into (2.6) we find

$$u(x,t) = \sum_{j=0}^{\infty} \frac{h^{(j)}(t)}{(2j)!} x^{2j}. \tag{2.7}$$

2. Absolute convergence. The expression (2.7) is a formal power series representation, which will be valid if we could show that the series converges absolutely. Let $\alpha > 1$ and define $h(t)$ as (see Figure 2.2)

$$h(t) = \begin{cases} e^{-t^{-\alpha}} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

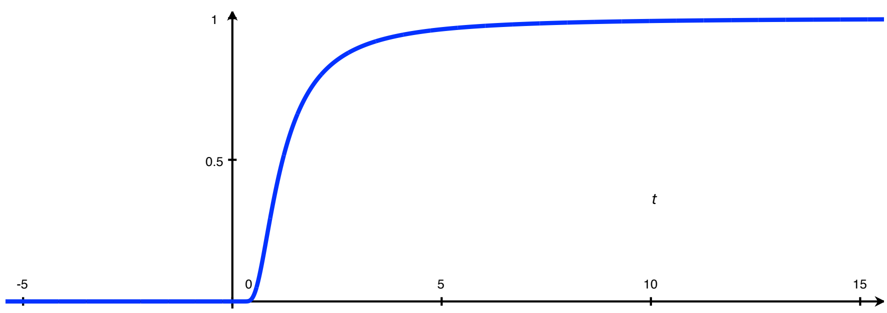


Figure 2.2: The function $h(t)$ (here with $\alpha = 2$).

It can be shown that there exists $\theta = \theta(\alpha) > 0$ such that for all $t > 0$

$$|h^{(j)}(t)| \leq \frac{j!}{(\theta t)^j} e^{-\frac{1}{2} t^{-\alpha}}$$

(see F. John, *Partial Differential Equations, Fourth Edition*, Chapter 3.3(c) Problem 3). Hence, the coefficient of x^{2j} in the power series satisfies

$$\frac{|h^{(j)|}(t)}{(2j)!} \leq \frac{j!}{(2j)!(\theta t)^j} e^{-\frac{1}{2}t^{-\alpha}}.$$

As $\frac{j!}{(2j)!} < \frac{1}{j!}$ we find that for $t > 0$ and any $x \in \mathbb{C}$,

$$\left| \sum_{j=0}^{\infty} \frac{h^{(j)}(t)}{(2j)!} x^{2j} \right| \leq \sum_{j=0}^{\infty} \frac{|x|^{2j}}{j!(\theta t)^j} e^{-\frac{1}{2}t^{-\alpha}} = e^{\frac{1}{t}\left(\frac{|x|^2}{\theta} - \frac{1}{2}t^{1-\alpha}\right)}, \tag{2.8}$$

since the exponential function has the well-known power series representation $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$ for any $z \in \mathbb{C}$. We have therefore proved that the power series representation of u appearing in (2.7) *converges pointwise* at every $x \in \mathbb{C}$ and $t > 0$, implying that it is a **valid power series representation**. Clearly, the convergence also holds for $t \leq 0$ where the series is trivial.

3. Uniform convergence. First, we note that from (2.8) it follows that for all x in a compact subset of $\mathbb{C}$, the limit $\lim_{t \downarrow 0} u(x,t) = 0$ holds uniformly. We only know that the power series representation of u is valid, but we need to show that the heat equation holds, i.e. we need to prove that we can differentiate the power series term-by-term, twice in x and once in t. For this we need uniform convergence. From (2.8), it follows that for $x \in \mathbb{C}, t \in \mathbb{R}$, the function

$$U(x,t) = \begin{cases} e^{\frac{1}{t}\left(\frac{x^2}{\theta} - \frac{1}{2}t^{1-\alpha}\right)} & t > 0, \\ 0 & t \leq 0, \end{cases}$$

serves as a uniform bound for the series expansion of $u(x,t)$. Since $U(x,t)$ is bounded uniformly for bounded complex x and all $t \in \mathbb{R}$, the series for u converges uniformly for bounded x and $t \in \mathbb{R}$. The same holds for all x and t derivatives of U and of the power series for u.

Therefore we can conclude that $u_t = u_{xx}$ and that $u \in C^\infty(\mathbb{R} \times \mathbb{R})$. □

2.6 The Backward Heat Equation is Ill-Posed

The 'backward' heat equation is the heat equation with time reversed, that is, it is the equation

$$u_t + \kappa u_{xx} = 0 \tag{2.9}$$

where $\kappa > 0$. In Theorem 2.3 we have seen that the heat equation takes initial data that is merely bounded and continuous and immediately produces a solution that is infinitely differentiable (in x and in t) for any $t > 0$. Intuitively, if we reverse time then we should '*un*-smooth' our initial data. In physics, this is translated into the notion of the *arrow of time*: there are certain phenomena that are not time-reversible; heat diffusion is the prime example of a process that cannot be reversed.

In this direction, there are various theorems that can be proved. Here we choose to use our previous results to show that we can take a sequence g_n of smooth initial data for the backward heat equation, which will have a short lifespan of $1/n$. That is, as $t \to \frac{1}{n}$, the solution will "blow-up".

Remark: There are two ways to think about the backward heat equation. Either reverse time in the heat equation, leading to the equation (2.9), or consider the forward heat equation (2.1) but with time starting from some $T > 0$ and going backwards. Below we choose the latter.

Theorem 2.6: Let $T > 0$. The backward heat equation, that is the equation

$$\begin{cases} u_t - \kappa u_{xx} = 0 & \text{in} \quad (x,t) \in \mathbb{R} \times (0,T), \\ u = g & \text{on} \quad (x,t) \in \mathbb{R} \times \{t = T\}, \end{cases} \tag{2.10}$$

with data prescribed at $t = T$, is ill-posed in the sense that there is a sequence $\{g_n\}_{n=1}^{\infty} \subset C^{\infty}(\mathbb{R})$ such that the corresponding unique physical solution u_n has a lifespan of $\frac{1}{n}$.

Proof. From Lemma 2.2 we know that the fundamental solution $\Phi(x,t)$ solves the heat equation $u_t - \kappa u_{xx} = 0$. Furthermore, as we have seen in Section 2.1, $\Phi(\cdot,t) \to \delta$ as $t \downarrow 0$ (in the sense of distributions). Therefore, $\Phi_n(x, t - T + \frac{1}{n})$ solves the backward equation with the 'initial' condition $g_n(x) = \Phi_n(x, \frac{1}{n})$ at time $t = T$ which is C^{∞}, but which as $t \downarrow T - \frac{1}{n}$ tends to δ (in the sense of distributions). □

Remark: We could have multiplied the g_n's by small positive constants to ensure that they are arbitrarily small in any reasonable norm, in which case they could be viewed as a perturbation of zero.

Chapter 3

Fourier Series: A Brief Review

A Fourier series is a series expansion of a function in terms of *sines* and/or *cosines*.

Joseph Fourier (1768-1830)

When studying heat flow, Joseph Fourier understood that the key was in writing a trigonometric series expansion of solutions, and studying them term by term. His 1807 paper on this subject was rejected as too imprecise, and was only published in 1822.

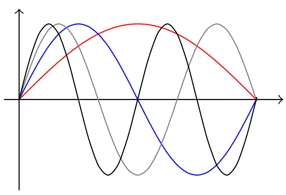
This section will be mostly formal, with some proofs verifying that the various series indeed converge in an appropriate sense.

3.1 Sine, Cosine and Full Fourier Series

3.1.1 Sine Series

Let $\phi(x)$ be some function on the interval $[0, L]$. We define its **sine series** to be

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$



Observing the all the sines in this series vanish at $x = 0$ and $x = L$, it is obvious that it would be advisable for ϕ to also vanish there. Indeed, this expansion is preferable for problems with *Dirichlet boundary conditions* where we shall require solutions to vanish on the boundary. Using the orthogonality of sines

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \,\mathrm{d}x = \begin{cases} 0 & n \neq m, \\ \frac{L}{2} & n = m, \end{cases}$$

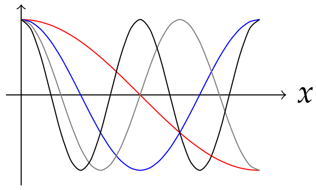
Exercise 3.1: Can you prove this?

we find that

$$B_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} \,\mathrm{d}x.$$

3.1.2 Cosine Series

Similarly, we can define the **cosine series** of a function $\phi(x)$ as



$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

Observing the *derivatives* of all the cosines in this series vanish at $x = 0$ and $x = L$, it is obvious that it would be advisable for ϕ to also have a derivative that vanishes there. Indeed, this expansion is preferable for problems with *Neumann boundary conditions* where we shall require the derivative of solutions to vanish on the boundary. Using the orthogonality of cosines

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} 0 & n \neq m, \\ \frac{L}{2} & n = m, \end{cases}$$

we find that

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x.$$

3.1.3 Full Fourier Series

We write the full Fourier series on the interval $(-L, L)$, where we have:

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right).$$

Using

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,\mathrm{d}x = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = 0$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\,\mathrm{d}x = \int_{-L}^{L} \cos\frac{n\pi x}{L}\,\mathrm{d}x = 0$$

we find that

$$A_n = \frac{1}{L}\int_{-L}^{L} \phi(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x, \quad \text{for} \quad n = 0, 1, 2, \ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} \phi(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x, \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Exercise 3.2: Find the coefficients of the Fourier sine and cosine series on $(0, L)$ and of the full Fourier series on $(-L, L)$ of

- $\phi(x) = 1$,
- $\phi(x) = x$.

3.1.4 Complex Form of the Fourier Series

Using the DeMoivre formulas

$$\sin\theta = \frac{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}}{2\mathrm{i}} \quad \text{and} \quad \cos\theta = \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2\mathrm{i}}$$

we find that we can replace the basis

$$\{1, \sin(\pi x/L), \cos(\pi x/L), \sin(2\pi x/L), \cos(2\pi x/L), \ldots\}$$

for functions on $(-L, L)$ with the elegant basis

$$\{e^{\frac{\mathrm{i}n\pi x}{L}}\}_{n\in\mathbb{Z}}$$

so that we can write

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{\mathrm{i}n\pi x}{L}}.$$

Using the fact that

$$\int_{-L}^{L} e^{\frac{\mathrm{i}n\pi x}{L}} e^{\frac{-\mathrm{i}m\pi x}{L}}\,\mathrm{d}x = \begin{cases} 0 & n \neq m, \\ 2L & n = m, \end{cases}$$

we find

$$c_n = \frac{1}{2L}\int_{-L}^{L} \phi(x) e^{\frac{-\mathrm{i}n\pi x}{L}}\,\mathrm{d}x.$$

3.2 Orthogonality and General Fourier Series

Consider now a general interval (a, b) and functions $f(x), g(x)$ (possibly complex-valued) defined on it. Define their **inner product** as:

$$(f, g) := \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x$$

and we say that f and g are **orthogonal** if $(f, g) = 0$.

Consider the following operators and boundary conditions:

- $\mathcal{L}_\mathrm{D}$ = negative second derivative operator with **Dirichlet** BCs:

$$\mathcal{L}_\mathrm{D} f(x) = -f''(x), \quad \text{and} \quad f(a) = f(b) = 0.$$

- $\mathcal{L}_\mathrm{N}$ = negative second derivative operator with **Neumann** BCs:

$$\mathcal{L}_\mathrm{N} f(x) = -f''(x), \quad \text{and} \quad f'(a) = f'(b) = 0.$$

- $\mathcal{L}_\mathrm{P}$ = negative second derivative operator with **periodic** BCs:

$$\mathcal{L}_\mathrm{P} f(x) = -f''(x), \quad \text{and} \quad f(a) = f(b),\ f'(a) = f'(b).$$

Lemma 3.1 (Green's Second Identity): Let $y_1, y_2 \in C^2(a,b)$, then

$$\int_a^b \left(-y_1''\overline{y_2} + y_1\overline{y_2}''\right) \mathrm{d}x = \left(-y_1'\overline{y_2} + y_1\overline{y_2}'\right)\Big|_{x=a}^{b}. \tag{3.1}$$

Proof. This is an exercise. □

Lemma 3.2: Assume that both y_1 and y_2 satisfy either Dirichlet, Neumann or periodic BCs. Then the RHS of (3.1) is 0.

Proof. This is an exercise. □

Observation: Let $\mathcal{L} \in \{\mathcal{L}_\mathrm{D}, \mathcal{L}_\mathrm{N}, \mathcal{L}_\mathrm{P}\}$ and let (λ, X) be an eigenvalue-eigenfunction pair of $\mathcal{L}$:

$$\mathcal{L}X = \lambda X \quad \text{in} \quad (a,b). \tag{3.2}$$

Then $(\lambda^*, \overline{X})$ is also an eigenvalue-eigenfunction pair of $\mathcal{L}$.[1]

Proof. This is an exercise. □

Theorem 3.3: Let $\mathcal{L} \in \{\mathcal{L}_\mathrm{D}, \mathcal{L}_\mathrm{N}, \mathcal{L}_\mathrm{P}\}$. Then $\mathcal{L}$ has no complex eigenvalues, and any eigenfunction can be taken to be real-valued.

Proof. In Green's Second Identity (3.1) replace both y_1 and y_2 with some function $X(x)$ that is an eigenfunction of $\mathcal{L}$ with eigenvalue λ. From Lemma 3.2 we know that the RHS of (3.1) is 0. Hence we have

$$\begin{aligned}
0 &= \int_a^b \left(-X''(x)\overline{X}(x) + X(x)\overline{X}''(x)\right) \mathrm{d}x \\
&= \int_a^b \left(\lambda X(x)\overline{X}(x) - \lambda^* X(x)\overline{X}(x)\right) \mathrm{d}x \\
&= (\lambda - \lambda^*) \int_a^b X(x)\overline{X}(x)\,\mathrm{d}x \\
&= (\lambda - \lambda^*) \int_a^b |X(x)|^2\,\mathrm{d}x.
\end{aligned}$$

Since $|X(x)|^2 \geq 0$ and since $X(x)$ is not trivial, the integral $\int_a^b |X(x)|^2\,\mathrm{d}x$ must be *strictly positive*. Then necessarily $\lambda - \lambda^* = 0$ which can only be true if $\lambda \in \mathbb{R}$.

It remains to be shown that the eigenfunction can be taken to be real-valued. Suppose that it is complex-valued and write it as

$$X(x) = Y(x) + \mathrm{i}Z(x)$$

where Y and Z are real-valued. Then

$$-Y''(x) - \mathrm{i}Z''(x) = -X''(x) = \lambda X(x) = \lambda Y(x) + \mathrm{i}\lambda Z(x).$$

Taking real and imaginary parts, we have

$$-Y''(x) = \lambda Y(x) \quad \text{and} \quad -Z''(x) = \lambda Z(x).$$

Furthermore, whichever boundary conditions are satisfied by X will also be satisfied by Y and by Z.

[1] We denote $\square^*$ the complex conjugate of a number, and by $\overline{\square}$ the complex conjugate of a function.

Exercise 3.3: Can you prove this?

Hence Y and Z are real-valued eigenfunctions with eigenvalue λ, satisfying the same boundary conditions. Observe that $\overline{X}$ has eigenvalue $\lambda^* = \lambda$ (all eigenvalues are real!). We conclude that we can replace X and $\overline{X}$ by Y and Z, observing that $\operatorname{span}(X, \overline{X}) = \operatorname{span}(Y, Z)$.

□

Theorem 3.4: Let $\mathcal{L} \in \{\mathcal{L}_\mathrm{D}, \mathcal{L}_\mathrm{N}, \mathcal{L}_\mathrm{P}\}$. Then any two eigenfunctions corresponding to different eigenvalues of $\mathcal{L}$ are orthogonal. Consequently, if any function is expressed in a series of these eigenfunctions, the coefficients are determined.

Proof. We already know that all eigenvalues are real and the all eigenfunctions can be taken to be real-valued. Take two eigenvalue-eigenfunction pairs (λ_1, X_1) and (λ_2, X_2) with $\lambda_1 \neq \lambda_2$. In (3.1) replace y_1 and y_2 by X_1 and X_2: the RHS is 0 by Lemma 3.2. So we have

$$\begin{aligned} 0 &= \int_a^b \left(-X_1''(x)X_2(x) + X_1(x)X_2''(x)\right) \mathrm{d}x \\ &= \int_a^b \left(\lambda_1 X_1(x)X_2(x) - \lambda_2 X_1(x)X_2(x)\right) \mathrm{d}x \\ &= (\lambda_1 - \lambda_2) \int_a^b X_1(x)X_2(x)\,\mathrm{d}x. \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ it follows that $\int_a^b X_1(x)X_2(x)\,\mathrm{d}x = 0$, that is X_1 and X_2 are orthogonal: $(X_1, X_2) = 0$.

Now, denote by X_n an eigenfunction corresponding to the eigenvalue λ_n and suppose that

$$\phi(x) = \sum_n A_n X_n(x) \tag{3.3}$$

is a convergent series[2]. Then

$$(\phi, X_m) = \left(\sum_n A_n X_n, X_m\right) = \sum_n A_n\,(X_n, X_m) = A_m (X_m, X_m)$$

because of the orthogonality, and therefore

$$A_m = \frac{(\phi, X_m)}{(X_m, X_m)}$$

is the formula for the coefficients. □

Remark: Note that a given eigenvalue might have two (or more) eigenfunctions corresponding to it that are not multiples of each other. These do not have to be orthogonal, but can be made to be orthogonal by the Gram-Schmidt procedure.

Theorem 3.5: Let $\mathcal{L} \in \{\mathcal{L}_\mathrm{D}, \mathcal{L}_\mathrm{N}, \mathcal{L}_\mathrm{P}\}$. Then $\mathcal{L}$ has no negative eigenvalues.

[2] We still need to discuss what 'convergence' precisely means.

Proof. We start with *Green's First Identity*:

$$\int_a^b f''(x)g(x)\,\mathrm{d}x = -\int_a^b f'(x)g'(x)\,\mathrm{d}x + f'(x)g(x)\Big|_{x=a}^{b} \tag{3.4}$$

(this is just integration by parts). Choose both f and g to be the same real eigenfunction X with real eigenvalue λ. Then:

$$\begin{aligned}
\text{LHS of (3.4)} &= \int_a^b X''(x)X(x)\,\mathrm{d}x = -\lambda \int_a^b X^2(x)\,\mathrm{d}x \\
\text{RHS of (3.4)} &= -\int_a^b X'(x)X'(x)\,\mathrm{d}x + X'(x)X(x)\Big|_{x=a}^{b} \\
&= -\int_a^b (X'(x))^2\,\mathrm{d}x + \underbrace{X'(b)X(b) - X'(a)X(a)}_{=0 \text{ because of BCs}}
\end{aligned}$$

It follows that

$$\lambda = \frac{\int_a^b (X'(x))^2\,\mathrm{d}x}{\int_a^b X^2(x)\,\mathrm{d}x} \geq 0.$$

Moreover, $\lambda = 0$ if and only if $X(x) \equiv \text{const} \neq 0$. Note that a nonzero constant can be an eigenfunction only for $\mathcal{L}_{\mathrm{N}}$ and $\mathcal{L}_{\mathrm{P}}$ but not $\mathcal{L}_{\mathrm{D}}$.

□

Theorem 3.6: Let $\mathcal{L} \in \{\mathcal{L}_{\mathrm{D}}, \mathcal{L}_{\mathrm{N}}, \mathcal{L}_{\mathrm{P}}\}$. Then $\mathcal{L}$ has infinitely many eigenvalues tending to $+\infty$.

Proof. Strictly speaking, this theorem is trivial, since it is easy to check that the eigenvalues in these cases are (up to a constant) $\lambda_n \sim n^2$ (see Chapter 4). However, this theorem holds for more general operators, and is therefore nontrivial. We do not include the proof here. A detailed proof can be found in W. A. Strauss, *Partial Differential Equations: An Introduction, Second Edition*, Chapter 11. □

3.3 Convergence of General Fourier Series

Our goal in this section is to understand, for an expression such as (3.3), in what sense the Fourier series converges. More generally, let f and $\{f_n\}_{n=1}^{\infty}$ be functions defined on $[a, b]$, and we want to consider whether the partial sum $S_N(x) := \sum_{n=1}^{N} f_n(x)$ converges to $f(x)$ as $N \to +\infty$. Let us recall three important notions of convergence in this case (we have already seen this in Section 2.5):

Three Notions of Convergence

We say that the series $\sum_{n=1}^{\infty} f_n(x)$ ...

converges pointwise to $f(x)$ in (a,b) if it converges for each $x \in (a,b)$. That is, for each $x \in (a,b)$ we have

$$\left|f(x) - S_N(x)\right| \to 0 \quad \text{as} \quad N \to +\infty;$$

converges uniformly to $f(x)$ in $[a,b]$ if

$$\max_{x\in[a,b]} \left|f(x) - S_N(x)\right| \to 0 \quad \text{as} \quad N \to +\infty;$$

converges in $\mathbf{L^2}$ (or *mean square*) to $f(x)$ in (a,b) if

$$\int_a^b \left|f(x) - S_N(x)\right|^2 \mathrm{d}x \to 0 \quad \text{as} \quad N \to +\infty.$$

Let $\mathcal{L} \in \{\mathcal{L}_\mathrm{D}, \mathcal{L}_\mathrm{N}, \mathcal{L}_\mathrm{P}\}$ and consider the eigenvalue problem (3.2). We know that $\mathcal{L}$ has infinitely many real and nonnegative eigenvalues

$$0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

and corresponding to them infinitely many eigenfunctions

$$X_1, X_2, X_3, \ldots$$

which can be taken to be real-valued and mutually orthogonal. The Fourier coefficients of the function f are defined as

$$A_n = \frac{(f, X_n)}{(X_n, X_n)} = \frac{\int_a^b f(x)\overline{X_n}(x)\,\mathrm{d}x}{\int_a^b |X_n(x)|^2\,\mathrm{d}x}.$$

We can now state three theorems regarding the three notions of convergence:

Theorem 3.7 (Uniform Convergence)**:** The Fourier series $\sum_n A_n X_n(x)$ converges to $f(x)$ *uniformly* on $[a,b]$ provided that

1. $f(x), f'(x)$ exist and are continuous on $[a,b]$, and
2. $f(x)$ satisfies that same boundary condition as $\mathcal{L}$.

Theorem 3.8 (L^2 Convergence)**:** The Fourier series $\sum_n A_n X_n(x)$ converges to $f(x)$ *in L^2* on (a,b) provided that $f(x)$ has finite L^2-norm, i.e.:

$$\int_a^b |f(x)|^2\,\mathrm{d}x < +\infty.$$

Theorem 3.9 (Pointwise Convergence)**:** 1. The classical Fourier series (sine, cosine or full) converges to $f(x)$ *pointwise* on (a, b) provided that $f(x)$ is a continuous function on $[a, b]$ and $f'(x)$ is piecewise continuous on $[a, b]$.

2. If $f(x)$ itself is only piecewise continuous on $[a, b]$ and $f'(x)$ is also piecewise continuous on $[a, b]$, then the Fourier series converges for all $x \in \mathbb{R}$ with

$$\sum_{n=1}^{\infty} A_n X_n(x) = \frac{f(x+) + f(x-)}{2} \quad \text{for all} \quad x \in (a, b).$$

The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $x \in \mathbb{R}$, where $f_{\text{ext}}(x)$ is the extension of $f(x)$ (odd periodic, even periodic or periodic).

Remark: Theorem 3.9 has several definitions which we have not encountered before. Let us clarify them:

1. For any function $f(x)$ we define

$$f(x+) := \lim_{\varepsilon \downarrow 0} f(x+\varepsilon) \quad \text{and} \quad f(x-) := \lim_{\varepsilon \downarrow 0} f(x-\varepsilon)$$

whenever these limits exist. These limits are the same at points of continuity of f.

2. Somewhat informally, the various extensions of a function $f(x)$ defined on (a, b) are as follows:
   - A **periodic extension** is a function $f_{\text{ext}}(x)$ defined on $\mathbb{R}$ that is obtained by copying and pasting f onto the intervals $(b, 2b-a)$, $(2b-a, 3b-2a), \ldots$ to the right, and similarly to the left.
   - An **odd extension** is a function $f_{\text{ext}}(x)$ obtained by first extending f to $(2a-b, a)$ oddly (around a), and then extending the resulting functions periodically.
   - An **even extension** is a function $f_{\text{ext}}(x)$ obtained by first extending f to $(2a-b, a)$ evenly (around a), and then extending the resulting functions periodically.

Since Theorem 3.9 is quite lengthy to state, it is easier to state it already for a function with period $2L$ defined on the whole real line. The following is an equivalent formulation of Theorem 3.9:

Theorem 3.9′ (Pointwise Convergence, equivalent form)**:** Let $f(x)$ be a $2L$-periodic function on $\mathbb{R}$. Assume that $f(x)$ and $f'(x)$ are both piecewise continuous. Then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for all $x \in \mathbb{R}$.

3.3.1 L^2 Theory: Bessel's Inequality and Parseval's Equality

The goal of this section is to prove Theorem 3.8 on the L^2 convergence of the Fourier series and provide the functional background required. The proof won't be complete

(Theorem 3.12 is given without proof). As we have seen, the inner product is defined as:

$$(f, g) := \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x.$$

This allows us to define a **norm**:

$$\|f\| = \sqrt{(f, f)} = \sqrt{\int_a^b |f(x)|^2\,\mathrm{d}x}$$

which leads to the notion of a distance (**metric**):

$$\|f - g\| = \sqrt{\int_a^b |f(x) - g(x)|^2\,\mathrm{d}x}.$$

Theorem 3.10 (Bessel's inequality)**:** Let $f(x)$ be a function defined on (a, b) with Fourier series $\sum_{n=1}^{\infty} A_n X_n(x)$. Then

$$\sum_{n=1}^{\infty} A_n^2 \|X_n\|^2 \leq \|f\|^2.$$

Proof. Since $f(x)$ is given as $f(x) = \sum_{n=1}^{\infty} A_n X_n(x)$, it would be sensible to split this expression into a partial sum and a *tail*:

$$f(x) = \sum_{n=1}^{\infty} A_n X_n(x) = \underbrace{\sum_{n=1}^{N} A_n X_n(x)}_{S_N(x)} + \sum_{n=N+1}^{\infty} A_n X_n(x)$$

so that the tail satisfies

$$\sum_{n=N+1}^{\infty} A_n X_n(x) = f(x) - S_N(x).$$

Define the *error* as

$$E_N := \left\| \sum_{n=N+1}^{\infty} A_n X_n \right\|^2 = \|f - S_N\|^2.$$

Then we have

$$\begin{aligned}
E_N &= \|f - S_N\|^2 \\
&= \int_a^b \left| f(x) - \sum_{n=1}^N A_n X_n(x) \right|^2 \mathrm{d}x \\
&= \int_a^b |f(x)|^2 \,\mathrm{d}x - 2 \sum_{n=1}^N \int_a^b f(x) A_n X_n(x) \,\mathrm{d}x + \sum_{n=1}^N \sum_{m=1}^N \int_a^b A_n A_m X_n(x) X_m(x) \,\mathrm{d}x \\
&= \|f\|^2 - 2 \sum_{n=1}^N A_n (f, X_n) + \sum_{n=1}^N \sum_{m=1}^N A_n A_m (X_n, X_m) \\
&= \|f\|^2 - 2 \sum_{n=1}^N A_n^2 \|X_n\|^2 + \sum_{n=1}^N A_n^2 \|X_n\|^2 \\
&= \|f\|^2 - \sum_{n=1}^N A_n^2 \|X_n\|^2.
\end{aligned}$$

Recall that E_N is, by definition, a square of a norm of some function, and therefore nonnegative. It follows that

$$\sum_{n=1}^N A_n^2 \|X_n\|^2 \le \|f\|^2.$$

But this is true for every $N \in \mathbb{N}$. Hence all partial sums $\sum_{n=1}^N A_n^2 \|X_n\|^2$ are *uniformly bounded*, which implies that we may take the limit $N \to +\infty$ to get

$$\sum_{n=1}^\infty A_n^2 \|X_n\|^2 \le \|f\|^2.$$

This is known as **Bessel's inequality**. □

Theorem 3.11: The Fourier series of f converges to f in L^2 if and only if there is an *equality* in Bessel's inequality.

Proof. By definition, S_N converges to f in the L^2 sense if and only if $E_N = \int_a^b |f(x) - S_N(x)|^2 \,\mathrm{d}x \to 0$ as $N \to +\infty$. However, the previous proof shows that $E_N \to 0$ as $N \to +\infty$ if and only if $\|f\|^2 - \sum_{n=1}^N A_n^2 \|X_n\|^2 \to 0$ as $N \to +\infty$. This last limit holds if and only if

$$\sum_{n=1}^\infty A_n^2 \|X_n\|^2 = \|f\|^2.$$

This is known as **Parseval's equality**. □

Complete set

The set of mutually orthogonal functions $\{X_i(x)\}_{i=1}^\infty$ is called **complete** if Parseval's equality is true for every f with finite L^2-norm: $\|f\|^2 = \int_a^b |f(x)|^2 \,\mathrm{d}x < +\infty$.

Theorem 3.8 states that the Fourier series converges in L^2 if f has finite L^2-norm. This can now equivalently be stated as follows:

Theorem 3.12 (L^2 Convergence, new version)**:** The eigenfunctions $\{X_i(x)\}_{i=1}^{\infty}$ coming from $\mathcal{L}_{\mathrm{D}}$, $\mathcal{L}_{\mathrm{N}}$ and $\mathcal{L}_{\mathrm{P}}$ form a complete set.

We do not prove this theorem here.

3.3.2 Uniform Convergence

Proof of Theorem 3.7. Here we prove Theorem 3.7 on the uniform convergence of the Fourier series whenever

1. $f(x), f'(x)$ exist and are continuous on $[a, b]$, and
2. $f(x)$ satisfies that same boundary condition as $\mathcal{L}$.

We prove for the case of the full Fourier series on $(-L, L)$ with periodic BCs. To simplify further, we take $L = \pi$. We write the series for f

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big(A_n\cos(nx) + B_n\sin(nx)\Big).$$

and the series for f' (we cannot assume that it is simply given by the term-by-term differentiation of the series for f)

$$f'(x) = \frac{1}{2}\widetilde{A}_0 + \sum_{n=1}^{\infty}\Big(\widetilde{A}_n\cos(nx) + \widetilde{B}_n\sin(nx)\Big).$$

Then for every $n = 0, 1, \ldots$

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\
\text{(int. by parts)} &= \underbrace{\frac{1}{n\pi}f(x)\sin(nx)\Big|_{x=-\pi}^{\pi}}_{=0} - \underbrace{\frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x}_{=-\frac{1}{n}\widetilde{B}_n} \\
&= -\frac{1}{n}\widetilde{B}_n
\end{aligned}$$

Similarly we find that

$$B_n = \frac{1}{n}\widetilde{A}_n.$$

Note that this step used the periodicity and continuity of f and f'. We therefore have:

$$\begin{aligned}
\sum_{n=1}^{\infty}\Big(|A_n\cos(nx)| + |B_n\sin(nx)|\Big) &\le \sum_{n=1}^{\infty}\Big(|A_n| + |B_n|\Big) = \sum_{n=1}^{\infty}\frac{1}{n}\Big(|\widetilde{A}_n| + |\widetilde{B}_n|\Big) \\
\text{Cauchy-Schwarz} &\le \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\Big(|\widetilde{A}_n| + |\widetilde{B}_n|\Big)^2\right)^{\frac{1}{2}} \\
&\le C\left(\sum_{n=1}^{\infty}2\Big(|\widetilde{A}_n|^2 + |\widetilde{B}_n|^2\Big)\right)^{\frac{1}{2}}
\end{aligned}$$

which is finite by Bessel's inequality. Hence

$$\sum_{n=1}^{\infty}\Big(|A_n\cos(nx)| + |B_n\sin(nx)|\Big) < +\infty,$$

that is, the Fourier series of f converges *absolutely*. Therefore, computing the difference between f and the partial Fourier sums, we find

$$\begin{aligned}
\max_{x\in[-\pi,\pi]}\left|f(x) - \frac{1}{2}A_0 - \sum_{n=1}^{N}\Big(A_n\cos(nx) + B_n\sin(nx)\Big)\right|
&= \max_{x\in[-\pi,\pi]}\left|\sum_{n=N+1}^{\infty}\Big(A_n\cos(nx) + B_n\sin(nx)\Big)\right| \\
&\le \max_{x\in[-\pi,\pi]}\sum_{n=N+1}^{\infty}\Big|A_n\cos(nx) + B_n\sin(nx)\Big| \\
&\le \sum_{n=N+1}^{\infty}(|A_n| + |B_n|)
\end{aligned}$$

which is the tail of a convergent series, i.e. it tends to 0 as $N \to +\infty$. By definition, this means that the Fourier series of f converges to f *uniformly*. $\square$

Chapter 4

Separation of Variables: The Wave and Heat Equations on an Interval

In this section we solve the initial boundary value problem (IBVP) for the wave and heat equations on an interval. We shall see that because of the simple geometry (an interval $(0, L)$ in the spatial variable) we can separate the two variable (spatial and temporal) and solve for each separately. It's important to note that this strategy won't work in more complex geometries, and it is therefore wrong to assume that this is a general strategy for solving PDEs!

Strategy

Make the ansatz that the solution $u(x, t)$ is the product of a function of x and a function of t:

$$u(x, t) = X(x)T(t). \tag{4.1}$$

4.1 Dirichlet Boundary Conditions

We start with homogeneous Dirichlet problems, where the values at the boundary of the various problems are fixed and equal to 0.

4.1.1 The Wave Equation

We start by considering the Dirichlet IBVP for the homogeneous wave equation:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in} \quad (x,t) \in (0,L) \times (0,+\infty), \\ u(0,t) = u(L,t) = 0 & \text{for} \quad t > 0, \\ u(x,0) = \phi(x) & \text{in} \quad x \in (0,L), \\ u_t(x,0) = \psi(x) & \text{in} \quad x \in (0,L). \end{cases} \tag{4.2}$$

This models a vibrating string of length L, fixed (at the same height) at the end-points, with initial position given by the function ϕ and the initial vertical speed given by ψ. Plugging in the ansatz (4.1) into the wave equation we have:

$$X(x)T''(t) - c^2 X''(x)T(t) = 0.$$

Dividing by c^2XT, and rearranging the equality, this becomes

$$-\frac{1}{c^2}\frac{T''}{T} = -\frac{X''}{X}. \tag{4.3}$$

Remark: Note the *negative* sign: considering the results of Section 3.2, and in particular Theorem 3.5, we anticipate that the operator $\mathcal{L}f = -f''$ is the natural one to consider.

In (4.3), the LHS is a function solely of t, while the RHS is a function solely of x. The only way for functions of two different *independent* variables to equal one another, is if they are both constant. We call this constant λ (again in anticipation of using the eigenvalue results of Section 3.2):

$$\underbrace{-\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \lambda = \underbrace{-\frac{X''}{X}}_{\text{function of } x}.$$

This leads to the two ordinary differential equations (ODEs) in x and in t:

$$-T''(t) = \lambda c^2 T(t) \quad \text{and} \quad -X''(x) = \lambda X(x).$$

From Theorem 3.5 we know that $\lambda \geq 0$, so that there exists $\beta \in \mathbb{R}$ such that $\beta^2 = \lambda$. We therefore replace λ by β^2.

Spatial part

In the spatial variable we have a Dirichlet problem on the interval $(0, L)$, which is exactly the operator $\mathcal{L}_\mathrm{D}$ from Section 3.2. Our goal now is to find *eigenvalue-eigenfunction pairs* of this problem. The equation for the spatial part is

$$\begin{cases} -X''(x) = \beta^2 X(x) & \text{in} \quad 0 < x < L, \\ \quad X(0) = X(L) = 0. \end{cases} \tag{4.4}$$

The equation is solved by sines and cosines, with the general solution having the form:

$$X(x) = C\cos\beta x + D\sin\beta x$$

where C and D are constants. We now **impose the boundary conditions** which will restrict that range of possible values of these constants. First, imposing $X(0) = 0$ we find

$$X(0) = 0 \quad \Rightarrow \quad C\underbrace{\cos 0}_{=1} + D\underbrace{\sin 0}_{=0} = 0 \quad \Rightarrow \quad \mathbf{C = 0}.$$

Now we impose $X(L) = 0$:

$$X(L) = 0 \quad \Rightarrow \quad D\sin\beta L = 0 \quad \Rightarrow \quad \boldsymbol{\beta L = n\pi} \qquad (n = 1, 2, \ldots).$$

Hence the range of possible values of β is *discretized* (sometimes also called *quantized*): $\beta_n = \frac{n\pi}{L}$, $n = 1, 2, \ldots$ We therefore conclude that the **eigenvalue-eigenfunction pair of the problem (4.4)** is

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \sin\frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

Temporal part

In the temporal variable we have the problem

$$-T''(t) = \beta^2 c^2 T(t), \qquad t > 0.$$

The general solution is

$$T(t) = A\cos\beta ct + B\sin\beta ct$$

where A and B are constants.

Back to the problem (4.2)

Plugging into the ansatz (4.1) the expressions we've obtained for $X(x)$ and for $T(t)$, we have

$$u_n(x,t) = \Big(A_n\cos\Big(\frac{n\pi}{L}ct\Big) + B_n\sin\Big(\frac{n\pi}{L}ct\Big)\Big)\sin\Big(\frac{n\pi x}{L}\Big)$$

By linearity, any finite sum of such solutions will still be a solution. That is,

$$u(x,t) = \sum_n \Big(A_n\cos\Big(\frac{n\pi}{L}ct\Big) + B_n\sin\Big(\frac{n\pi}{L}ct\Big)\Big)\sin\Big(\frac{n\pi x}{L}\Big) \tag{4.5}$$

is also a solution of the problem (4.2), where we have thus far only imposed the boundary conditions, i.e. u as defined in (4.5) satisfies $u(0,t) = u(L,t) = 0$. We are now ready to **impose the initial conditions**:

First we have

$$\begin{aligned}\phi(x) = u(x,0) &= \sum_n \Big(A_n\underbrace{\cos\Big(\frac{n\pi}{L}c\cdot 0\Big)}_{=1} + B_n\underbrace{\sin\Big(\frac{n\pi}{L}c\cdot 0\Big)}_{=0}\Big)\sin\Big(\frac{n\pi x}{L}\Big)\\ &= \sum_n A_n\sin\Big(\frac{n\pi x}{L}\Big).\end{aligned}$$

Since the sum appearing in (4.5) is finite, we can take a time derivative, to get

$$u_t(x,t) = \sum_n \Big(-A_n\frac{n\pi}{L}c\sin\Big(\frac{n\pi}{L}ct\Big) + B_n\frac{n\pi}{L}c\cos\Big(\frac{n\pi}{L}ct\Big)\Big)\sin\Big(\frac{n\pi x}{L}\Big).$$

Then we obtain

$$\begin{aligned}\psi(x) = u_t(x,0) &= \sum_n \Big(-A_n\frac{n\pi}{L}c\underbrace{\sin\Big(\frac{n\pi}{L}c\cdot 0\Big)}_{=0} + B_n\frac{n\pi}{L}c\underbrace{\cos\Big(\frac{n\pi}{L}c\cdot 0\Big)}_{=1}\Big)\sin\Big(\frac{n\pi x}{L}\Big)\\ &= \sum_n B_n\frac{n\pi}{L}c\sin\Big(\frac{n\pi x}{L}\Big).\end{aligned}$$

We therefore have **Fourier sine series** expansions for both ϕ and ψ, for which we know the formula for the coefficients (see Section 3.1.1): from the series for ϕ we have

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x \tag{4.6}$$

and from the series for ψ we have

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{4.7}$$

Thus, (4.5), (4.6) and (4.7) together provide us with the general form of the solution of (4.2) (under the various assumptions we've made along the way, of course). Moreover, there's no claim about this being *the* solution. We've merely shown that this is *a* solution.

4.1.2 The Heat (Diffusion) Equation

Here we consider the Dirichlet IBVP for the homogeneous heat equation:

$$\begin{cases} u_t(x,t) - \kappa u_{xx}(x,t) = 0 & \text{in} \quad (x,t) \in (0,L)\times(0,+\infty), \\ u(0,t) = u(L,t) = 0 & \text{for} \quad t > 0, \\ u(x,0) = \phi(x) & \text{in} \quad x \in (0,L). \end{cases} \tag{4.8}$$

This models a conductive rod of length L, with an initial temperature $\phi(x)$ at the point x, with the ends always held at 0 degrees. Plugging in the ansatz (4.1) into the heat equation we have:

$$X(x)T'(t) - \kappa X''(x)T(t) = 0.$$

Dividing by κXT, and rearranging the equality, this becomes

$$-\frac{1}{\kappa}\frac{T'}{T} = -\frac{X''}{X}.$$

As before, we find that a function of x is identically equal to a function of t; this can only be possible if these two functions are constant. We call this constant λ. This leads to the two ordinary differential equations (ODEs) in x and in t:

$$-T'(t) = \lambda\kappa T(t) \quad \text{and} \quad -X''(x) = \lambda X(x).$$

Considering the equation for X, Theorem 3.5 implies that $\lambda \geq 0$, and we let $\beta^2 = \lambda$.

Spatial part

The spatial part is identical to the spatial part in Section 4.1.1 and we again find the **eigenvalue-eigenfunction pairs**:

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \sin\frac{n\pi x}{L}, \qquad n = 1, 2, \dots$$

Temporal part

The temporal part differs from the temporal part in the wave equation. Here we find the *first order* ODE

$$-T'(t) = \lambda\kappa T(t), \qquad t > 0,$$

where the general solution is

$$T(t) = Ae^{-\lambda\kappa t}$$

where A is a constant.

Back to the problem (4.8)

As before, we plug these results into the ansatz (4.1) to obtain

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t} \sin\left(\frac{n\pi x}{L}\right),$$

and by linearity, any sum of such solutions will still be a solution, so

$$u(x,t) = \sum_n A_n e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t} \sin\left(\frac{n\pi x}{L}\right) \tag{4.9}$$

is also a solution. As before, we **impose the initial condition**:

$$\phi(x) = u(x,0) = \sum_n A_n \underbrace{e^{-\left(\frac{n\pi}{L}\right)^2 \kappa \cdot 0}}_{=1} \sin\left(\frac{n\pi x}{L}\right) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right).$$

We again have a **Fourier sine series** expansion for ϕ and, for which the coefficients are given by the formula (see Section 3.1.1):

$$A_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{4.10}$$

The equations (4.9) and (4.10) together provide us with the general form of the solution of (4.8).

4.2 Neumann Boundary Conditions

After considering the Dirichlet problems, we can also consider the equivalent homogeneous Neumann problems, where the derivatives at the boundary is fixed and equal to 0. The majority of the analysis is the same as in the Dirichlet case, except for the imposition of the boundary conditions for the spatial variable. This will lead to Fourier *cosine* series.

4.2.1 The Wave Equation

The Nueumann IBVP for the wave equation models a vibrating string (lying horizontally) of length L which at both ends is free to move transversally along a track with no resistance (so that the angle of the string at the track has no vertical component):

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{in} \quad (x,t) \in (0,L) \times (0,+\infty), \\ u_x(0,t) = u_x(L,t) = 0 & \text{for} \quad t > 0, \\ u(x,0) = \phi(x) & \text{in} \quad x \in (0,L), \\ u_t(x,0) = \psi(x) & \text{in} \quad x \in (0,L). \end{cases} \tag{4.11}$$

As before, plugging in the ansatz (4.1) into the wave equation we have:

$$X(x)T''(t) - c^2 X''(x)T(t) = 0.$$

Dividing by c^2XT, and rearranging, we have, as before

$$\underbrace{-\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \lambda = \underbrace{-\frac{X''}{X}}_{\text{function of } x}.$$

This leads to the two ordinary differential equations (ODEs) in x and in t:

$$-T''(t) = \lambda c^2 T(t) \quad \text{and} \quad -X''(x) = \lambda X(x).$$

From Theorem 3.5 we know that $\lambda \geq 0$, so that there exists $\beta \in \mathbb{R}$ such that $\beta^2 = \lambda$. We therefore replace λ by β^2.

Spatial part

Now we consider the Neumann problem on the interval $(0, L)$, which is the operator $\mathcal{L}_{\mathrm{N}}$ from Section 3.2. Our goal now is to find *eigenvalue-eigenfunction pairs* of this problem. The equation for the spatial part is

$$\begin{cases} -X''(x) = \beta^2 X(x) & \text{in} \quad 0 < x < L, \\ \quad X'(0) = X'(L) = 0. \end{cases} \tag{4.12}$$

The equation is solved by sines and cosines, with the general solution having the form:

$$X(x) = C\cos\beta x + D\sin\beta x \quad \text{so that} \quad X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x$$

where C and D are constants. We now **impose the boundary conditions** which will restrict that range of possible values of these constants. First, imposing $X'(0) = 0$ we find

$$X'(0) = 0 \quad \Rightarrow \quad -C\beta\underbrace{\sin 0}_{=0} + D\beta\underbrace{\cos 0}_{=1} = 0 \quad \Rightarrow \quad \mathbf{D = 0}.$$

Now we impose $X'(L) = 0$:

$$X'(L) = 0 \quad \Rightarrow \quad C\beta\sin\beta L = 0 \quad \Rightarrow \quad \boldsymbol{\beta L = n\pi} \qquad (n = 1, 2, \ldots).$$

Hence the range of possible values of β is: $\beta_n = \frac{n\pi}{L}$, $n = 1, 2, \ldots$ We therefore conclude that the **eigenvalue-eigenfunction pair of the problem (4.12)** is

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \cos\frac{n\pi x}{L}, \qquad n = 0, 1, 2, \ldots$$

where we note that 0 **is also an eigenvalue**, as opposed to the Dirichlet case!

Temporal part

In the temporal variable we have the problem

$$-T''(t) = \beta^2 c^2 T(t), \qquad t > 0.$$

For $\beta > 0$ the general solution is

$$T(t) = A\cos\beta ct + B\sin\beta ct$$

where A and B are constants, and for $\beta = 0$ we get

$$T(t) = A + Bt.$$

Back to the problem (4.11)

Plugging into the ansatz (4.1) the expressions we've obtained for $X(x)$ and for $T(t)$, we have

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right)\right)\cos\left(\frac{n\pi x}{L}\right)$$

By linearity, any sum of such solutions will still be a solution. We must now also include the solution corresponding to the 0 eigenvalue, which is just $\frac{1}{2}A_0 + \frac{1}{2}B_0 t$ (the $\frac{1}{2}$ factor is a choice we make for normalisation). That is,

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_n \left(A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right)\right)\cos\left(\frac{n\pi x}{L}\right) \tag{4.13}$$

is also a solution of the problem (4.11), where we have thus far only imposed the boundary conditions, i.e. u as defined in (4.13) satisfies $u_x(0,t) = u_x(L,t) = 0$. We are now ready to **impose the initial conditions**:

First we have

$$\begin{aligned}\phi(x) = u(x,0) &= \frac{1}{2}A_0 + \frac{1}{2}B_0 \cdot 0 + \sum_n \left(A_n \underbrace{\cos\left(\frac{n\pi}{L}c\cdot 0\right)}_{=1} + B_n \underbrace{\sin\left(\frac{n\pi}{L}c\cdot 0\right)}_{=0}\right)\cos\left(\frac{n\pi x}{L}\right)\\ &= \frac{1}{2}A_0 + \sum_n A_n \cos\left(\frac{n\pi x}{L}\right).\end{aligned}$$

Since the sum appearing in (4.13) is finite, we can take a time derivative, to get

$$u_t(x,t) = \frac{1}{2}B_0 + \sum_n \left(-A_n \frac{n\pi}{L}c \sin\left(\frac{n\pi}{L}ct\right) + B_n \frac{n\pi}{L}c\cos\left(\frac{n\pi}{L}ct\right)\right)\cos\left(\frac{n\pi x}{L}\right).$$

Then we obtain

$$\begin{aligned}\psi(x) = u_t(x,0) &= \frac{1}{2}B_0 + \sum_n \left(-A_n \frac{n\pi}{L}c \underbrace{\sin\left(\frac{n\pi}{L}c\cdot 0\right)}_{=0} + B_n \frac{n\pi}{L}c \underbrace{\cos\left(\frac{n\pi}{L}c\cdot 0\right)}_{=1}\right)\cos\left(\frac{n\pi x}{L}\right)\\ &= \frac{1}{2}B_0 + \sum_n B_n \frac{n\pi}{L}c\cos\left(\frac{n\pi x}{L}\right).\end{aligned}$$

We therefore have **Fourier cosine series** expansions for both ϕ and ψ, for which we know the formula for the coefficients (see Section 3.1.2): from the series for ϕ we have

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n = 0,1,2,\dots \tag{4.14}$$

and from the series for ψ we have

$$B_0 = \frac{2}{L}\int_0^L \psi(x)\,\mathrm{d}x,$$

and

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n = 1,2,\dots \tag{4.15}$$

Thus, (4.13), (4.14) and (4.15) together provide us with the general form of the solution of (4.11) (under the various assumptions we've made along the way, of course).

4.2.2 The Heat (Diffusion) Equation

The Neumann IBVP for the heat equation is

$$\begin{cases} u_t(x,t) - \kappa u_{xx}(x,t) = 0 & \text{in} \quad (x,t) \in (0,L) \times (0,+\infty), \\ u_x(0,t) = u_x(L,t) = 0 & \text{for} \quad t > 0, \\ u(x,0) = \phi(x) & \text{in} \quad x \in (0,L). \end{cases} \tag{4.16}$$

This models a conductive rod of length L, with an initial temperature $\phi(x)$ at the point x, with the insulated ends. Plugging in the ansatz (4.1) into the heat equation we have:

$$X(x)T'(t) - \kappa X''(x)T(t) = 0.$$

Dividing by κXT, and rearranging the equality, this becomes

$$-\frac{1}{\kappa}\frac{T'}{T} = -\frac{X''}{X}.$$

As before, we find that a function of x is identically equal to a function of t; this can only be possible if these two functions are constant. We call this constant λ. This leads to the two ordinary differential equations (ODEs) in x and in t:

$$-T'(t) = \lambda\kappa T(t) \quad \text{and} \quad -X''(x) = \lambda X(x).$$

Considering the equation for X, Theorem 3.5 implies that $\lambda \geq 0$, and we let $\beta^2 = \lambda$.

Spatial part

The spatial part is identical to the spatial part in Section 4.2.1 and we again find the **eigenvalue-eigenfunction pairs**:

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \cos\frac{n\pi x}{L}, \qquad n = 0, 1, 2, \ldots$$

where we note that 0 **is also an eigenvalue**, as opposed to the Dirichlet case!

Temporal part

As in Section 4.1.2 we find the *first order* ODE

$$-T'(t) = \lambda\kappa T(t), \qquad t > 0,$$

where the general solution for $\lambda > 0$ is

$$T(t) = Ae^{-\lambda\kappa t}$$

and for $\lambda = 0$ it is

$$T(t) = A$$

where A is a constant.

Back to the problem (4.16)

As before, we plug these results into the ansatz (4.1) to obtain

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t} \cos\left(\frac{n\pi x}{L}\right),$$

and by linearity, any sum of such solutions will still be a solution (including the solution corresponding to the 0 eigenvalue), so

$$u(x,t) = \frac{1}{2}A_0 + \sum_n A_n e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t} \cos\left(\frac{n\pi x}{L}\right) \tag{4.17}$$

is also a solution. As before, we **impose the initial condition**:

$$\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_n A_n \underbrace{e^{-\left(\frac{n\pi}{L}\right)^2 \kappa \cdot 0}}_{=1} \cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}A_0 + \sum_n A_n \cos\left(\frac{n\pi x}{L}\right).$$

We again have a **Fourier cosine series** expansion for ϕ and, for which the coefficients are given by the formula (see Section 3.1.1):

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n = 0,1,2,\ldots \tag{4.18}$$

The equations (4.17) and (4.18) together provide us with the general form of the solution of (4.16).

Exercise 4.1:

1. A quantum mechanical particle on the line with an infinite potential outside the interval $(0,L)$ is given by Schrödinger's equation $u_t = iu_{xx}$ in $(0,L)$ with homogeneous Dirichlet conditions at the ends. Separate the variable and write the solution in series form.

2. Consider a metal rod with ends at $x = 0, L$ which is insulated along its length but not at its two ends. Initially, its temperature is identically 1, but then it is immediately plunged into an ice bath at both its ends, where the temperature is fixed at 0 for all $t > 0$. Write the PDE, initial and boundary conditions, and write a formula for the temperature $u(x,t)$ at later times. You may use the following infinite series expansion:

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right).$$

3. Solve the diffusion problem $u_t = \kappa u_{xx}$ in $0 < x < L$ with the mixed boundary conditions $u(0,t) = u_x(L,t) = 0$.

4. Solve the wave equation $u_{tt} = c^2 u_{xx}$ in $0 < x < L$ with the mixed boundary conditions $u_x(0,t) = u(L,t) = 0$.

Chapter 5

The Laplace Equation in a Bounded Domain

The goal of this chapter is to study the Dirichlet problem for the Laplace equation in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \tag{5.1}$$

The precise regularity required of both $\partial\Omega$ and f will be specified later. This boundary value problem (BVP) is one of the most fundamental in all the natural sciences, and a brief discussion was presented in Section 1.7.2. We will prove that solutions exist (via the so-called *Perron's method*) and are unique. As part of the proof, we will encounter many important aspects of this problem, such as the *maximum principle*, *Gauss' law of arithmetic mean*, *Poisson's formula* and the notion of *subharmonic functions*.

5.1 Basic Properties

We start with some basic properties of the Laplacian and of the equation (5.1) which shall be useful in the sequel.

5.1.1 Uniqueness of Solutions

Establishing that solutions to (5.1) (if they exist) are unique is actually quite straightforward.

Proposition 5.1 (Uniqueness of Solutions)**:** The boundary value problem (5.1) has at most one solution in $C^2(\overline{\Omega})$.

Proof. The starting point is Green's identities in dimension d. The general form of these identities (which we saw in 1D in (3.4) and (3.1)) is:

Green's Identities

Assume that $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is an open and bounded set for which the divergence theorem holds (for instance, $\partial\Omega \in C^1$ suffices). Let n be the unit normal vector on $\partial\Omega$ oriented to the exterior of Ω. Let $u, v \in C^2(\overline{\Omega})$. Then

$$\int_\Omega v\,\Delta u\,\mathrm{d}x = -\int_\Omega \sum_{i=1}^d v_{x_i} u_{x_i}\,\mathrm{d}x + \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,\mathrm{d}S, \tag{5.2}$$

$$\int_\Omega v\,\Delta u\,\mathrm{d}x = \int_\Omega u\,\Delta v\,\mathrm{d}x + \int_{\partial\Omega}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)\mathrm{d}S. \tag{5.3}$$

By choosing particular u and v we can get additional useful identities. Choosing $u = v$ in (5.2) leads to an energy identity:

$$\int_\Omega \sum_{i=1}^d u_{x_i}^2\,\mathrm{d}x + \int_\Omega u\,\Delta u\,\mathrm{d}x = \int_{\partial\Omega} u\frac{\partial u}{\partial n}\,\mathrm{d}S.$$

It follows that if $\Delta u = 0$ in Ω and either $u = 0$ on $\partial\Omega$ or $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, then

$$\int_\Omega \sum_{i=1}^d u_{x_i}^2\,\mathrm{d}x = 0.$$

Since the integrand is nonnegative, it follows that $u_{x_i} = 0$ in Ω for all $i = 1, \ldots, d$, so that u must be constant in Ω. Now, suppose that (5.1) has two solutions u_1 and u_2. Define $u = u_1 - u_2$. Then u solves

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

so that u must be constant in Ω. Since $u = 0$ on $\partial\Omega$, that constant must be 0. So $0 \equiv u = u_1 - u_2$, hence $u_1 \equiv u_2$. $\square$

Exercise 5.1: Can you show that solutions to the Neumann problem are unique only *up to a constant*?

5.1.2 Fundamental Solution and Radial Symmetry

In Section 2.1 we saw the fundamental solution for the heat equation in $(x, t) \in \mathbb{R} \times (0, +\infty)$. There is a similar notion for the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad \mathbb{R}^d.$$

In spherical coordinates, let $r = \sqrt{x_1^2 + \cdots x_d^2}$ be the radial variable in $\mathbb{R}^d$, then we define:

The Fundamental Solution

The function

$$\Psi(r) = \begin{cases} \frac{\log r}{2\pi} & d = 2, \\ \frac{r^{2-d}}{(2-d)\omega_d} & d \geq 3, \end{cases}$$

is called the **fundamental solution** of Laplace's equation in $\mathbb{R}^d$, where $r > 0$ and where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{1}{2}d)}$ is the surface area of the unit sphere in $\mathbb{R}^d$.

[1] We immediately see that $\Psi'(r) = \omega_d^{-1} r^{-(d-1)}$ and that

$$\Delta_x \Psi = \Psi''(r) + \frac{d-1}{r}\Psi'(r) = 0 \quad \text{for} \quad r > 0,$$

that is Ψ solves the Laplace equation in $\mathbb{R}^d \setminus \{0\}$. In the case of the fundamental solution of the heat equation we have seen that there's an intimate relationship with the Dirac δ distribution. This is the case here too, where Ψ satisfies

$$\Delta_x \Psi = \delta$$

in the sense of distributions. As before, we do not elaborate on this, as the theory of distributions is beyond the scope of this course.

Observe that our radial coordinate r can be generalized: instead of being the distance from the origin, we could have taken it to be the distance from some arbitrary fixed point $\xi \in \mathbb{R}^d$, that is

$$r = \sqrt{(x_1 - \xi_1)^2 + \cdots (x_d - \xi_d)^2}. \tag{5.4}$$

Then the discussion from above still holds: we can still define the fundamental solution $\Psi(r)$ and the formal equation

$$\Delta_x \Psi = \delta_\xi$$

still holds, where now the δ distribution is centered around ξ, that is $\delta_\xi(x) = \delta(x - \xi)$.

The Kernel

The fundamental solution is therefore a **kernel** depending on x and ξ, which we write as:

$$K(x, \xi) = \Psi(|x - \xi|).$$

[1]The gamma function evaluated at half-integers is given by $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{1}{2} + n) = \binom{n - \frac{1}{2}}{n} n! \sqrt{\pi}$ and at the integers by $\Gamma(n) = (n-1)!$

5.1.3 Analyticity of Harmonic Functions

Proposition 5.2: Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $u \in C^2(\overline{\Omega})$ be harmonic: $\Delta u = 0$ in Ω. Then u is real analytic in Ω.[2]

Proof. Fix $\xi \in \Omega$. The starting point is Green's second identity (5.3), with $v(x) = \Psi(r)$ where r is as defined in (5.4). Since v is singular at $x = \xi$ we cut out a small ball around ξ: define $B(\xi, \rho) = \{y \in \mathbb{R}^d \mid |y - \xi| < \rho\}$ the ball of radius ρ around ξ and

$$\Omega_\rho = \Omega \setminus B(\xi, \rho).$$

We choose ρ small enough so that $B(\xi, \rho) \subset \Omega$. The boundary of Ω_ρ has two components:

$$\partial\Omega_\rho = \partial\Omega \cup \partial B(\xi, \rho).$$

Since v is harmonic in B_ρ, (5.3) becomes

$$\int_{\Omega_\rho} v \,\Delta u \,\mathrm{d}x = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \mathrm{d}S + \underbrace{\int_{\partial B(\xi,\rho)} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \mathrm{d}S}_{I} \tag{5.5}$$

where n is the unit normal pointing outward from Ω_ρ, that is in $B(\xi, \rho)$ the vector n points *toward* ξ. Since v has been chosen to be the fundamental solution, we have that $v = \Psi(\rho)$ and $\frac{\partial v}{\partial n} = -\Psi'(\rho)$ on $\partial B(\xi, \rho)$. Thus:

$$\begin{aligned} I &= \int_{\partial B(\xi,\rho)} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \mathrm{d}S \\ &= \Psi(\rho) \int_{\partial B(\xi,\rho)} \frac{\partial u}{\partial n} \,\mathrm{d}S + \Psi'(\rho) \int_{\partial B(\xi,\rho)} u \,\mathrm{d}S \\ &= \underbrace{-\Psi(\rho) \int_{B(\xi,\rho)} \Delta u \,\mathrm{d}x}_{I_A} + \underbrace{\omega_d^{-1} \rho^{-(d-1)} \int_{\partial B(\xi,\rho)} u \,\mathrm{d}S}_{I_B}. \end{aligned}$$

The first part

$$I_A = -\Psi(\rho) \int_{B(\xi,\rho)} \Delta u \,\mathrm{d}x \approx -\Delta u(\xi) \Psi(\rho) \mathrm{vol}(B(\xi, \rho))$$

tends to 0 as $\rho \to 0$ since Δu is continuous at ξ and since $\mathrm{vol}(B(\xi, \rho)) \sim \rho^d$ tends to 0 faster than the rate with which $|\Psi(\rho)|$ tends to $+\infty$.

Exercise 5.2: Prove that $I_A \to 0$ as $\rho \to 0$ rigorously (as above, this can be proved even without using the fact that u is harmonic).

The second part has the limit

$$I_B = \omega_d^{-1} \rho^{-(d-1)} \int_{\partial B(\xi,\rho)} u \,\mathrm{d}S \approx \omega_d^{-1} \rho^{-(d-1)} u(\xi) \omega_d \rho^{d-1} = u(\xi)$$

since u is continuous at ξ.

[2] Recall that a function $f(x)$ is called *real analytic* if it is C^∞ and its Taylor series converges pointwise to $f(x)$ for all x in a neighborhood of every point x_0 in its domain.

Exercise 5.3: Prove that $I_B \to u(\xi)$ as $\rho \to 0$ rigorously.

It therefore follows that as $\rho \to 0$, the expression (5.5) tends to

$$\int_\Omega v \, \Delta u \, \mathrm{d}x = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \mathrm{d}S + u(\xi) \tag{5.6}$$

We now recall that u is harmonic, which implies that

$$u(\xi) = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \mathrm{d}S \tag{5.7}$$

where we recall that v has been chosen as $v(x) = \Psi(r)$, with Ψ centered around an arbitrary $\xi \in \Omega$.

In (5.7) we can take as many derivatives in ξ as we would like: v depends on x and on ξ and is infinitely differentiable in both so long as $x \neq \xi$. This implies that $u \in C^\infty$. By continuing x and ξ into a complex neighborhood, the same argument implies that u is real analytic. □

5.1.4 Gauss' Law of Arithmetic Mean

Proposition 5.3 (Gauss' Law of Arithmetic Mean)**:** Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let u be harmonic in Ω. Then

$$u(\xi) = \frac{1}{\omega_d \rho^{d-1}} \int_{|x-\xi|=\rho} u(x) \, \mathrm{d}S_x$$

where $\xi \in \mathbb{R}^d$ and $\rho > 0$ are such that $B(\xi, \rho) \subset \Omega$. **That is, for a harmonic function u in a closed ball, the value of u at the center equals the average of the values of u on the surface.**

Proof. Let us rewrite (5.6) in terms of the kernel K (despite the fact that u is harmonic, we keep the term Δu until the last moment):

$$u(\xi) = \int_\Omega K(x,\xi) \, \Delta u \, \mathrm{d}x + \int_{\partial\Omega} \left(u(x) \frac{\partial K(x,\xi)}{\partial n_x} - K(x,\xi) \frac{\partial u(x)}{\partial n_x} \right) \mathrm{d}S_x$$

Let w be another harmonic function in Ω and define

$$G(x,\xi) = K(x,\xi) + w(x).$$

This is also a kernel with pole ξ, and can therefore replace the kernel K in the above expression:

$$u(\xi) = \int_\Omega G(x,\xi) \, \Delta u \, \mathrm{d}x + \int_{\partial\Omega} \left(u(x) \frac{\partial G(x,\xi)}{\partial n_x} - G(x,\xi) \frac{\partial u(x)}{\partial n_x} \right) \mathrm{d}S_x. \tag{5.8}$$

Exercise 5.4: Verify that we can indeed replace K with G.

Let $\rho > 0$ and choose

$$\Omega = B(\xi, \rho) \quad \text{and} \quad w(x) = -\Psi(\rho)$$

so that

$$G(x, \xi) = K(x, \xi) - \Psi(\rho) = \Psi(|x - \xi|) - \Psi(\rho).$$

On $\partial\Omega$, i.e. for x such that $|x - \xi| = \rho$, G satisfies

$$G = 0 \quad \text{and} \quad \frac{\partial G(x, \xi)}{\partial n_x} = \Psi'(\rho) = \omega_d^{-1} \rho^{-(d-1)}$$

so that (5.8) becomes

$$u(\xi) = \int_{|x-\xi|<\rho} (\Psi(|x - \xi|) - \Psi(\rho)) \Delta u(x) \, dx + \omega_d^{-1} \rho^{-(d-1)} \int_{|x-\xi|=\rho} u(x) \, dS_x. \tag{5.9}$$

Recalling that u is harmonic, this becomes

$$u(\xi) = \frac{1}{\omega_d \rho^{d-1}} \int_{|x-\xi|=\rho} u(x) \, dS_x$$

and the proof is complete. □

5.1.5 Subharmonic Functions

Considering (5.9) and noting that Ψ is an increasing function of its argument, we find that if $\Delta u(x) \geq 0$ in the ball $|x - \xi| \leq \rho$, then

$$u(\xi) \leq \frac{1}{\omega_d \rho^{d-1}} \int_{|x-\xi|=\rho} u(x) \, dS_x. \tag{5.10}$$

Subharmonic Functions

A function $u \in C(\Omega)$ is called **subharmonic** if for each $\xi \in \Omega$ if (5.10) holds for all ρ sufficiently small. We denote the set of subharmonic functions in Ω by $\sigma(\Omega)$.

Observe that any function $u \in C^2(\Omega)$ with $\Delta u \geq 0$ is subharmonic.

Subharmonic functions will prove to be important in the sequel.

5.2 The Maximum Principle

5.2.1 The Weak Maximum Principle

Theorem 5.4 (The Weak Maximum Principle)**:** Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and assume that $\Delta u \geq 0$ in Ω. Then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \tag{5.11}$$

Proof. First we note that since $\partial\Omega \subset \overline{\Omega}$, necessarily $\max_{\overline{\Omega}} u \geq \max_{\partial\Omega} u$. Hence we need to prove that $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$.

Let us start with the case of a strict inequality: suppose that a function v satisfies $\Delta v > 0$ in Ω. Suppose, by contradiction, that $\max_{\overline{\Omega}} v > \max_{\partial\Omega} v$. Then there exists some $\xi \in \Omega$ such that $v(\xi) = \max_{\overline{\Omega}} v$. The point ξ is a local maximum, so $v_{x_k}(\xi) = 0$ and $v_{x_k x_k}(\xi) \leq 0$ for all $k = 1, \ldots, d$. But this means that $\sum_k v_{x_k x_k}(\xi) \leq 0$, contradicting the assumption that $\Delta v > 0$ in Ω. Hence $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$.

Returning to the function u: for any $\varepsilon > 0$, consider the auxiliary function $v = u + \varepsilon|x|^2$. Since $\Delta |x|^2 > 0$, it follows that $\Delta v > 0$ (strict inequality). Then by the preceding argument, $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$. This implies:

$$\begin{aligned}
\max_{\overline{\Omega}} u + \min_{\overline{\Omega}} \varepsilon|x|^2 &\leq \max_{\overline{\Omega}}(u + \varepsilon|x|^2) \\
&= \max_{\overline{\Omega}} v \\
&= \max_{\partial\Omega} v \\
&= \max_{\partial\Omega}(u + \varepsilon|x|^2) \leq \max_{\partial\Omega} u + \max_{\partial\Omega} \varepsilon|x|^2
\end{aligned}$$

This is true for any $\varepsilon > 0$, so it follows that $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$ as required. □

Corollary 5.5: 1. If in Theorem 5.4 we assume that $\Delta u = 0$, then (5.11) holds also for $-u$, and since $\min u = -\max(-u)$ we obtain

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

2. Furthermore, using the fact that for $q \in \mathbb{R}$ the absolute value satisfies $|q| = \max(q, -q)$, we have that if $\Delta u = 0$ then

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Corollary 5.6: In Theorem 5.4 assume that $\Delta u = 0$. Then if $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.

Consequently, we can strengthen Proposition 5.1 to not require derivatives on $\partial\Omega$:

Proposition 5.7 (Stronger Uniqueness Theorem)**:** The boundary value problem (5.1) has at most one solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$.

5.2.2 The Strong Maximum Principle

We can now prove a stronger version of the maximum principle:

Theorem 5.8 (The Strong Maximum Principle)**:** Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected. Let $u \in C(\Omega)$ be subharmonic in Ω. Then either u is constant in Ω, or

$$u(\xi) < \sup_{\Omega} u \quad \text{for all} \quad \xi \in \Omega.$$

Proof. Let $M = \sup_\Omega u \in \mathbb{R}\cup\{+\infty\}$ and decompose Ω into the disjoint union $\Omega = \Omega_1 \sqcup \Omega_2$ where

$$\Omega_1 = \{\xi \in \Omega \mid u(\xi) = M\} \quad \text{and} \quad \Omega_2 = \{\xi \in \Omega \mid u(\xi) < M\}.$$

Since u is continuous in Ω the set Ω_2 is open. We shall show that Ω_1 is also open. Let $\xi \in \Omega_1$. Then since u is subharmonic, from (5.10) we have that for small enough $\rho > 0$

$$\begin{aligned} 0 &\le \int_{|x-\xi|=\rho} u(x)\,\mathrm{d}S_x - \omega_d \rho^{d-1} u(\xi) \\ &= \int_{|x-\xi|=\rho} (u(x) - u(\xi))\,\mathrm{d}S_x \\ &= \int_{|x-\xi|=\rho} (u(x) - M)\,\mathrm{d}S_x. \end{aligned}$$

By the definition of M, $u(x) - M \le 0$. Since u is continuous, so is $u(x) - M$. We therefore have that for every $\rho > 0$ sufficiently small, the integral of a non-positive continuous integrand is non-negative. The only way for this to be true is if the integrand is identically 0. This means that for all $\rho > 0$ sufficiently small, $u(x) = M$ for all x such that $|x - \xi| = \rho$. Therefore there is a neighborhood of ξ that belongs to Ω_1, implying that Ω_1 is open.

The set Ω is open, bounded and *connected*. As such, in cannot be decomposed into two disjoint open nonempty sets. It therefore follows that either Ω_1 or Ω_2 is empty. □

It immediately follows that:

Corollary 5.9: Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected. Let $u \in C^0(\overline{\Omega})$ be subharmonic in Ω. Then either u is constant in $\overline{\Omega}$, or

$$u(\xi) < \max_{\partial\Omega} u \quad \text{for all} \quad \xi \in \Omega.$$

5.3 Integral Representation of Solutions

The goal here is to be able to derive a formula for the values of a harmonic function in a domain, only using its values on the boundary. This is done in the spirit of Gauss' Law of Arithmetic Mean and the formula (5.8).

5.3.1 The Green's Function

In Section 5.1 we started with the kernel $K(x, \xi)$, from which we defined another kernel $G(x, \xi) = K(x, \xi) + w(x)$ where w was a harmonic function. Therefore kernels are not uniquely defined. However, among all kernels there is a special one called a *Green's function*:

Green's Function

A kernel $G(x,\xi)$ is called a **Green's function** if

$$G(x,\xi) = K(x,\xi) + v(x,\xi), \quad \text{where} \quad x \in \overline{\Omega},\ \xi \in \Omega,\ x \neq \xi,$$

if $v(\cdot,\xi) \in C^2(\overline{\Omega})$ satisfies $\Delta_x v = 0$ is such that

$$G(x,\xi) = 0 \quad \text{for} \quad x \in \partial\Omega,\ \xi \in \Omega.$$

Lemma 5.10: For any domain Ω there is a unique Green's function.

Proof. Suppose that G_1 and G_2 are two Green's functions, with

$$G_1(x,\xi) = K(x,\xi) + v_1(x,\xi) \quad \text{and} \quad G_2(x,\xi) = K(x,\xi) + v_2(x,\xi)$$

Then $G_1(x,\xi) - G_2(x,\xi) = v_1(x,\xi) - v_2(x,\xi)$. For any $\xi \in \Omega$, both v_1 and v_2 satisfy the equation

$$\begin{cases} \Delta_x v(\cdot,\xi) = 0 & \text{in} \quad \Omega, \\ v(\cdot,\xi) = -K(\cdot,\xi) & \text{on} \quad \partial\Omega. \end{cases} \tag{5.12}$$

However, from Proposition 5.1 we know that this problem has a unique solution, so that $v_1 = v_2$. □

In the expression (5.8), which we repeat here,

$$u(\xi) = \int_\Omega G(x,\xi)\,\Delta u\,\mathrm{d}x + \int_{\partial\Omega}\left(u(x)\frac{\partial G(x,\xi)}{\partial n_x} - G(x,\xi)\frac{\partial u(x)}{\partial n_x}\right)\mathrm{d}S_x,$$

if G is the Green's function and u is harmonic, we have

$$u(\xi) = \int_{\partial\Omega} u(x)\frac{\partial G(x,\xi)}{\partial n_x}\,\mathrm{d}S_x, \quad \text{where} \quad \xi \in \Omega.$$

So the values of u in Ω are uniquely determined by the values of u on $\partial\Omega$!

The Green's Function for a Ball

To explicitly write a formula for the Green's function of a particular domain Ω one would need an explicit solution to the problem (5.12) recalling that

$$K(x,\xi) = \Psi(|x-\xi|) = \begin{cases} \frac{\log|x-\xi|}{2\pi} & d = 2, \\ \frac{|x-\xi|^{2-d}}{(2-d)\omega_d} & d \geq 3. \end{cases}$$

This is generally impossible. However, when Ω possesses certain symmetries this becomes feasible. One example is when Ω is a ball of some radius $a > 0$ (without loss of generality, suppose that the ball is centered around the origin):

$$\Omega = B(0,a) = \{y \in \mathbb{R}^d \mid |y| < a\}.$$

Given $\xi \in B(0,a)$, define its **reflection** with respect to $\partial B(0,a)$:

$$\xi^* = \frac{a^2}{|\xi|^2}\xi$$

which is a point outside the ball: $\xi^* \in \mathbb{R}^d \setminus \overline{B}(0,a)$. The point ξ^* is such that

$$\frac{|x-\xi^*|}{|x-\xi|} = \frac{a}{|\xi|} = \text{const} \quad \text{for any} \quad x \in \partial B(0,a).$$

This can be seen by some elementary geometric considerations.

Proposition 5.11: The Green's function for the ball $B(0,a)$ is given by

$$\begin{aligned} G(x,\xi) &= K(x,\xi) - K\Big(\frac{a}{|\xi|}x, \frac{a}{|\xi|}\xi^*\Big) \\ &= \frac{1}{2\pi}\log\frac{|x-\xi|}{|x-\xi^*|} - \frac{1}{2\pi}\log\frac{a}{|\xi|} \end{aligned}$$

for $d = 2$ and by

$$\begin{aligned} G(x,\xi) &= K(x,\xi) - \left(\frac{|\xi|}{a}\right)^{2-d} K(x,\xi^*) \\ &= \frac{1}{(2-d)\omega_d}\left(|x-\xi|^{2-d} - \left(\frac{|\xi|}{a}\right)^{2-d}|x-\xi^*|^{2-d}\right) \end{aligned}$$

for $d \geq 3$.

Proof. This is an exercise. □

5.3.2 Poisson's Integral Formula

Lemma 5.12: Let $\Omega = B(0,a) = \{y \in \mathbb{R}^d \mid |y| < a\}$ be the open ball of radius $a > 0$ centered around the origin and let $G(x,\xi)$ be the associated Green's function, as defined in Proposition 5.11. Then for $\xi \in \Omega$ and $x \in \partial\Omega$,

$$\frac{\partial G(x,\xi)}{\partial n_x} = \frac{1}{a\omega_d}\frac{a^2 - |\xi|^2}{|x-\xi|^d}$$

Proof. This is an exercise. □

The Poisson Kernel and Poisson's Integral Formula

The function

$$H(x,\xi) := \frac{\partial G(x,\xi)}{\partial n_x} = \frac{1}{a\omega_d}\frac{a^2-|\xi|^2}{|x-\xi|^d}$$

is called the **Poisson kernel**. For u harmonic inside $\Omega = B(0,a)$, from (5.8) **Poisson's integral formula** follows:

$$u(\xi) = \int_{\partial\Omega} H(x,\xi)u(x)\,\mathrm{d}S_x \quad \text{for} \quad \xi \in \Omega. \tag{5.13}$$

The Poisson kernel satisfies several important properties:

Properties of the Poisson Kernel

1. For all $x \in \overline{\Omega}$ and $\xi \in \Omega$ with $x \neq \xi$, $H(x,\xi) \in C^\infty$.
2. For all $x \in \partial\Omega$ and $\xi \in \Omega$, $\Delta_\xi H(x,\xi) = 0$.
3. For all $\xi \in \Omega$, $\int_{\partial\Omega} H(x,\xi)\,\mathrm{d}S_x = 1$.
4. For all $x \in \partial\Omega$ and $\xi \in \Omega$, $H(x,\xi) > 0$.
5. If $\zeta \in \partial\Omega$, then
$$\lim_{\substack{\xi\to\zeta \\ \xi\in\Omega}} H(x,\xi) = 0$$
uniformly in $x \in \Omega$ for $|x-\zeta| > \delta > 0$.

Verifying the properties of the Poisson kernel. Properties 1, 4 and 5 follow directly from the formula for H, and are left as an exercise.

Property 2. This can also be shown directly from the expression for H; alternatively, it follows from the facts that (i) $H(x,\xi) = \frac{\partial G(x,\xi)}{\partial n_x}$, (ii) $\Delta_x G(x,\xi) = 0$, and (iii) $G(x,\xi) = G(\xi,x)$.

Property 3. This follows from applying (5.13) to the function $u \equiv 1$. □

5.3.3 Constructing a Harmonic Function in a Ball from Boundary-Values

The preceding discussion immediately leads to the following result where u is constructed in Ω from given values f on $\partial\Omega$:

Theorem 5.13: Let $\Omega = B(0,a) = \{y \in \mathbb{R}^d \mid |y| < a\}$ be the open ball of radius $a > 0$ centered around the origin. Let f be continuous on $\partial\Omega$. Define $u(\xi)$ to be equal to f on $\partial\Omega$ and by the formula
$$u(\xi) = \int_{\partial\Omega} H(x,\xi) f(x)\,\mathrm{d}S_x \tag{5.14}$$
for $\xi \in \Omega$. Then $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

Proof. The proof relies on the five properties of H. Taking derivatives of $u(\xi)$ in (5.13), these can be commuted with the integration in the right hand side, then Properties 1 and 2 imply that $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

We therefore only need to show that $u \in C^0(\overline{\Omega})$, and specifically the continuity *at* the boundary. Fix $\zeta \in \partial\Omega$ and let $\xi \in \Omega$; we want to estimate the difference $u(\xi) - f(\zeta)$.

Using *Property 3*, we can write, for $\delta > 0$,

$$
\begin{aligned}
u(\xi) - f(\zeta) &= \int_{\partial\Omega} H(x,\xi)(f(x) - f(\zeta))\,\mathrm{d}S_x \\
&= \int_{\partial\Omega\cap B(\zeta,\delta)} H(x,\xi)(f(x) - f(\zeta))\,\mathrm{d}S_x \\
&\quad + \int_{\partial\Omega\cap B(\zeta,\delta)^C} H(x,\xi)(f(x) - f(\zeta))\,\mathrm{d}S_x =: I_1 + I_2.
\end{aligned}
$$

The term I_1. Since $f \in C^0(\partial\Omega)$, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that whenever $x \in B(\zeta,\delta) \cap \partial\Omega$, $|f(x) - f(\zeta)| < \varepsilon$. *Properties 3* and *4* consequently imply that $|I_1| < \varepsilon$.

The term I_2. Define $M = \max_{\partial\Omega} |f|$, let ε, δ be as above and let $\xi \in \Omega$. From *Properties 4* and *5* we know that $H(x,\xi) > 0$ for $x \in \partial\Omega$, and $H(x,\xi) \to 0$ uniformly in x as ξ approaches $\partial\Omega$. Therefore there exists $\delta' = \delta'(\varepsilon, \delta(\varepsilon)) > 0$ such that whenever $\xi \in \Omega \cap B(\zeta,\delta')$ and $x \in \partial\Omega \cap B(\zeta,\delta)^C$,

$$
H(x,\xi) < \frac{\varepsilon}{2M\omega_d a^{d-1}}
$$

(see Figure 5.1). It follows that also $|I_2| < \varepsilon$, and consequently

$$
|u(\xi) - f(\zeta)| < 2\varepsilon \quad \text{for} \quad \xi \in \Omega \cap B(\zeta,\delta')
$$

which shows that u is continuous at the boundary point ζ.

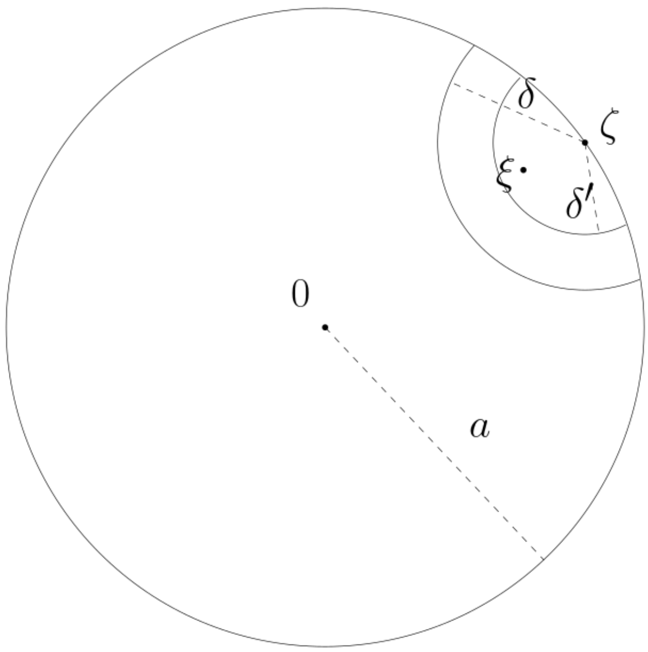


Figure 5.1: The construction in the proof of Theorem 5.13.

□

Exercise 5.5: Verify the details behind the estimates $|I_1| < \varepsilon$ and $|I_2| < \varepsilon$.

5.3.4 Completeness and Compactness of the Set of Harmonic Functions

The formula (5.14) leads to estimates on derivatives of harmonic functions which, in turn, allow us to show that the space of harmonic functions is complete.

Proposition 5.14: Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $u_k \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $k = 1, 2, \dots$, be a sequence of harmonic functions in Ω that converges uniformly on $\partial\Omega$ to some function f. Then u_k also converge uniformly in $\overline{\Omega}$ to some limit u. Moreover, u is harmonic in Ω (in particular, $u \in C^\infty(\Omega)$).

Proof. Note that since all the functions u_k are continuous on $\partial\Omega$, their uniform limit f is also continuous on $\partial\Omega$.

Claim: There exists $u \in C^0(\overline{\Omega})$ such that the sequence u_k converges to u uniformly in $\overline{\Omega}$.

Exercise 5.6: Prove the preceding claim.

To show that u is infinitely differentiable and harmonic we need to obtain uniform estimates of derivatives of the functions u_k on compact subsets of Ω. These will imply that all partial derivatives of the sequence u_k converge uniformly as well (albeit not on the boundary $\partial\Omega$). Let $\eta \in \Omega$ and let $\delta(\eta) = d(\eta, \partial\Omega)$ be the distance from η to the boundary of Ω, which is necessarily positive, $\delta(\eta) > 0$. Take some $a \in (0, \delta(\eta))$ and consider the ball $B(\eta, a) \subset \Omega$. Then (5.14) can be applied, and we have

$$u_k(\xi) = \int_{\partial B(\eta,a)} H(x,\xi) u_k(x) \, \mathrm{d}S_x \quad \text{for any} \quad \xi \in B(\eta, a).$$

Let $v_k(y) = u_k(y + \eta)$. Then the above expression becomes centered at the origin:

$$v_k(\zeta) = \int_{\partial B(0,a)} H(x,\zeta) v_k(x) \, \mathrm{d}S_x \quad \text{for any} \quad \zeta \in B(0, a).$$

Taking a derivative with respect to ζ_i and plugging in $\zeta = 0$, one finds

$$\partial_{\zeta_i} v_k(0) = \int_{\partial B(0,a)} \partial_{\zeta_i} H(x,\zeta)\big|_{\zeta=0} v_k(x) \, \mathrm{d}S_x = \frac{d}{\omega_d a^{d+1}} \int_{\partial B(0,a)} x_i v_k(x) \, \mathrm{d}S_x$$

Exercise 5.7: Verify the above computation.

Returning to u_k, the evaluation at $\zeta = 0$ is now replaced with $\xi = \eta$ and we have

$$\begin{aligned} \partial_{\xi_i} u_k(\eta) &= \frac{d}{\omega_d a^{d-1}} \int_{\partial B(0,a)} x_i u_k(x + \eta) \, \mathrm{d}S_x \\ &= \frac{d}{\omega_d a^{d-1}} \int_{\partial B(\eta,a)} (x_i - \eta_i) u_k(x) \, \mathrm{d}S_x. \end{aligned}$$

Estimating, we find

$$|\partial_{\xi_i} u_k(\eta)| \le \frac{d}{a} \max_{\partial B(\eta,a)} |u_k(x)|.$$

Exercise 5.8: Verify the above estimate.

Since the sequence u_k converges uniformly to u in $\overline{\Omega}$, there exists N such that for all $k \geq N$, $\max_{\overline{\Omega}} |u_k| \leq 2 \max_{\overline{\Omega}} |u|$ so that we have the uniform bound

$$|\partial_{\xi_i} u_k(\eta)| \leq \frac{2d}{a} \max_{\partial B(\eta,a)} |u(x)|.$$

The above estimate is true for any $a \in (0, \delta(\eta))$, and, in particular, by letting $a \to \delta(\eta)$ we deduce the estimate

$$|\partial_{\xi_i} u_k|(\eta) \leq \frac{2d}{\delta(\eta)} \max_{\overline{\Omega}} |u| \quad \text{for all} \quad \eta \in \Omega,\ k \geq N,\ i = 1, \ldots, d.$$

One can easily see that we could have taken as many mixed partial derivatives of u_k as we wish, and their resulting estimate would still be in terms of $\max_{\overline{\Omega}} |u|$, with the numerical constant in front changing (and always depending continuously on η), though potentially becoming unbounded as η approaches the boundary $\partial\Omega$. Therefore, on any compact subset $K \subset \Omega$ any mixed partial derivatives of u_k converge uniformly to the same mixed partial derivatives of u (here we use the fact that $\delta(\eta)$ is bounded uniformly away from 0 for any $\eta \in K$). It follows that u is harmonic in Ω and belongs to $C^\infty(\Omega)$. □

5.4 Existence of Solutions for the Dirichlet Problem (Perron's Method)

We now are finally ready to solve the problem (5.1), which we restate here,

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \quad u = f & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is open, bounded and connected.[3] The key to solving this problem is the use of subharmonic functions. We remind the reader that we denote by $\sigma(\Omega)$ the set of all subharmonic functions in Ω.

Idea of the Proof

To show that the problem (5.1) has a solution we will look at the set

$$\left\{ v \in \sigma(\Omega) \cap C^0(\overline{\Omega}) \quad \middle| \quad v \leq f \text{ on } \partial\Omega \right\}$$

and show that the supremum over all such functions v is the desired solution.

This proof, due to Oscar Perron (1880-1975), is exactly 100 years old (as of the writing these notes): published in December 1923! [4]

[3] Ω will also be required to satisfy the *barrier postulate*, see below. Eventually Ω will not be required to be connected, this is only a requirement coming from the maximum principle.

[4] O. Perron. Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$. *Math. Z.* **18**, 42-54 (1923).

Notation: For brevity, we denote the arithmetic mean of a function u on $\partial B(\xi,\rho)$ by $M_u(\xi,\rho)$:

$$M_u(\xi,\rho) = \frac{1}{\omega_d \rho^{d-1}} \int_{|x-\xi|=\rho} u(x)\,\mathrm{d}S_x$$

Then the definition of subharmonicity means that a function $u \in \sigma(\Omega)$ satisfies $u(\xi) \le M_u(\xi,\rho)$ for all $\xi \in \Omega$ and all $\rho > 0$ sufficiently small.

Exercise 5.9: Prove that the set $\sigma(\Omega)$ is closed under addition.

Exercise 5.10: Prove that a harmonic function is also subharmonic.

Definition 5.15: Let $u \in C^0(\Omega)$ and let $\xi \in \Omega$ and $\rho > 0$ be such that $\overline{B}(\xi,\rho) \subset \Omega$. Define $u_{\xi,\rho} \in C^0(\Omega)$ as the function that is obtained from u by replacing u inside $B(\xi,\rho)$ by a harmonic function:

$$\begin{cases} u_{\xi,\rho} = u & \text{in} \quad \Omega \setminus B(\xi,\rho), \\ \Delta\, u_{\xi,\rho} = 0 & \text{in} \quad B(\xi,\rho). \end{cases}$$

There exists such a function by Theorem 5.13, since we only need to find a harmonic function in the ball $B(\xi,\rho)$. Uniqueness follows from Proposition 5.7. Hence $u_{\xi,\rho}$ is well-defined.

5.4.1 Preliminary Lemmas

Lemma 5.16: For $u \in \sigma(\Omega)$ and $\overline{B}(\xi,\rho) \subset \Omega$ we have

$$u(x) \le u_{\xi,\rho}(x) \quad \text{for all} \quad x \in \Omega,$$

and $u_{\xi,\rho} \in \sigma(\Omega)$.

Proof. The definition of $u_{\xi,\rho}$ implies that $u = u_{\xi,\rho}$ in $\Omega \setminus B(\xi,\rho)$. So we only need to consider the set $B(\xi,\rho)$. In this set, u is subharmonic and $u_{\xi,\rho}$ is harmonic. Since $u_{\xi,\rho}$ is harmonic there, so is $-u_{\xi,\rho}$. Using Exercise 5.9 we see that

$$u - u_{\xi,\rho} \in \sigma(B(\xi,\rho)).$$

Additionally,

$$\begin{cases} u - u_{\xi,\rho} & \text{belongs to} \quad C^0(\overline{B}(\xi,\rho)), \\ u - u_{\xi,\rho} & \text{vanishes on} \quad \partial B(\xi,\rho). \end{cases}$$

The Weak Maximum Principle (Theorem 5.4) implies that $u - u_{\xi,\rho} \le 0$ in $B(\xi,\rho)$. We conclude that $u \le u_{\xi,\rho}$ in Ω.

It is left to show that $u_{\xi,\rho} \in \sigma(\Omega)$. This requires us to show that for every $\zeta \in \Omega$ there exists $\tau_0 = \tau_0(\zeta) > 0$ such that for all $\tau \in (0,\tau_0)$, $u_{\xi,\rho}(\zeta) \le M_{u_{\xi,\rho}}(\zeta,\tau)$. This is true for $\zeta \in \Omega \setminus \overline{B}(\xi,\rho)$, where $u_{\xi,\rho} = u$ and u is subharmonic. It is also true for $\zeta \in B(\xi,\rho)$ where $u_{\xi,\rho}$ is harmonic. So it only remains to be shown for $\zeta \in \partial B(\xi,\rho)$. For such ζ we know that $u_{\xi,\rho}(\zeta) = u(\zeta)$ and that $u(\zeta) \le M_u(\zeta,\tau)$ for all $\tau > 0$ sufficiently small since $u \in \sigma(\Omega)$. However, since $u \le u_{\xi,\rho}$ in Ω (we have just proven this), it follows that $M_u(\zeta,\tau) \le M_{u_{\xi,\rho}}(\zeta,\tau)$. Therefore we have that $u_{\xi,\rho}(\zeta) \le M_{u_{\xi,\rho}}(\zeta,\tau)$ for $\zeta \in \partial B(\xi,\rho)$ and for all $\tau > 0$ sufficiently small, which completes the proof. □

Lemma 5.17: Let $u \in \sigma(\Omega)$. For any $\xi \in \Omega$ and *any* $\rho > 0$ such that $\overline{B}(\xi, \rho) \subset \Omega$, the inequality $u(\xi) \leq M_u(\xi, \rho)$ holds.

Remark: Note that, by definition, any $u \in \sigma(\Omega)$ satisfies $u(\xi) \leq M_u(\xi, \rho)$ for all $\rho > 0$ *sufficiently small*. In this lemma, however, we need to prove that this inequality holds for *all* $\rho > 0$ such that $\overline{B}(\xi, \rho) \subset \Omega$.

Proof. Let $u \in \sigma(\Omega)$ and $\overline{B}(\xi, \rho) \subset \Omega$. From Lemma 5.16 we know that $u(x) \leq u_{\xi,\rho}(x)$ for all $x \in \Omega$, and, in particular, $u(\xi) \leq u_{\xi,\rho}(\xi)$. Since $u_{\xi,\rho}$ is defined to be harmonic in $B(\xi, \rho)$ it follows that $u_{\xi,\rho}(\xi) = M_{u_{\xi,\rho}}(\xi, \rho)$ from Gauss' Law of Arithmetic Mean (Proposition 5.3). But since $u_{\xi,\rho} = u$ on $\partial B(\xi, \rho)$, it follows that $M_{u_{\xi,\rho}}(\xi, \rho) = M_u(\xi, \rho)$. □

Lemma 5.18: A function u is harmonic in Ω if and only if $u, -u \in \sigma(\Omega)$.

Proof. Direction $\Rightarrow$: If u is harmonic then so is $-u$. The conclusion follows from Exercise 5.10.

Direction $\Leftarrow$: Assume that $u, -u \in \sigma(\Omega)$. Then by Lemma 5.16, for any $\overline{B}(\xi, \rho) \subset \Omega$,

$$u(x) \leq u_{\xi,\rho}(x) \quad \text{and} \quad -u(x) \leq -u_{\xi,\rho}(x) \quad \text{for all} \quad x \in \Omega,$$

so that $u(x) = u_{\xi,\rho}(x)$ for all $x \in \Omega$, implying that u is harmonic in Ω. □

Lemma 5.19: Let $u \in C^0(\Omega)$ and suppose that for any $\xi \in \Omega$ there exists $\rho_0 = \rho_0(\xi) > 0$ such that for all $\rho \in (0, \rho_0)$ the equality $u(\xi) = M_u(\xi, \rho)$ holds. Then u is harmonic in Ω.

Proof. This is left as an exercise. □

5.4.2 Construction of the Candidate Solution

Candidate Solution to the Dirichlet Problem

Let $f \in C^0(\partial\Omega)$. Define the set

$$\sigma_f(\overline{\Omega}) = \left\{ v \in \sigma(\Omega) \cap C^0(\overline{\Omega}) \;\middle|\; v \leq f \text{ on } \partial\Omega \right\}$$

and for any $x \in \Omega$ the function

$$w_f(x) = \sup_{v \in \sigma_f(\overline{\Omega})} v(x).$$

This function will (later) be our candidate solution to (5.1).

Lemma 5.20: Define

$$m_* = \min_{\partial\Omega} f \quad \text{and} \quad m^* = \max_{\partial\Omega} f.$$

Then $m_* \in \sigma_f(\overline{\Omega})$ (therefore $\sigma_f(\overline{\Omega})$ is not empty) and any $u \in \sigma_f(\overline{\Omega})$ satisfies $u(x) \leq m^*$ for any $x \in \overline{\Omega}$ (therefore $w_f(x)$ is well-defined).

Proof. This is left as an exercise. □

Lemma 5.21: Let $u_1, \ldots, u_k \in \sigma_f(\overline{\Omega})$ and for all $x \in \overline{\Omega}$ define $v(x) = \max_{1\leq j\leq k} u_j(x)$. Then $v \in \sigma_f(\overline{\Omega})$.

Proof. First we note that $v \in C^0(\overline{\Omega})$.

Exercise 5.11: Prove that $v \in C^0(\overline{\Omega})$.

So it remains to show that $v \in \sigma(\Omega)$ and that $v \leq f$ on $\partial\Omega$. This latter fact is obvious from the definition of v. To show subharmonicity, we use the fact that $u_1, \ldots, u_k$ are subharmonic to write (for all $\rho > 0$ sufficiently small):

$$v(\xi) = \max_{1\leq j\leq k} u_j(\xi) \leq \max_{1\leq j\leq k} M_{u_j}(\xi,\rho) \leq M_v(\xi,\rho).$$

Exercise 5.12: Prove the last inequality above.

This proves that $v \in \sigma(\Omega)$, and the proof is complete. □

As mentioned before, w_f is our candidate solution to (5.1). The following proposition proves that w_f is indeed harmonic. What will remain to be shown is that $w_f = f$ on $\partial\Omega$.

Proposition 5.22: The function w_f is harmonic in Ω.

Proof. Let $\xi \in \Omega$ and let $0 < \rho' < \rho$ be such that $B(\xi,\rho') \subset \overline{B}(\xi,\rho) \subset \Omega$. Let $\{x^k\}_{k=1}^{\infty} \subset B(\xi,\rho')$ be a sequence of points in $B(\xi,\rho')$. See Figure 5.2. By the definition of w_f, there are functions $\{u_k^j\}_{j,k=1}^{\infty} \subset \sigma_f(\overline{\Omega})$ such that

$$\lim_{j\to+\infty} u_k^j(x^k) = w_f(x^k) \quad \text{for every} \quad k \in \mathbb{N}, \tag{5.15}$$

where the convergence is *from below*. Define

$$u^j(x) = \max\left\{m_*, u_1^j(x), u_2^j(x), \ldots, u_j^j(x)\right\} \quad \text{for all} \quad x \in \overline{\Omega},\ j \in \mathbb{N}.$$

By Lemma 5.21, $u^j \in \sigma_f(\overline{\Omega})$ for all $j \in \mathbb{N}$. Moreover, $u^j(x) \geq u_k^j(x)$ for all $x \in \overline{\Omega}$ and $j \geq k$ by definition. We can sandwich u^j between u_k^j ($k \leq j$) and w_f, which means that (5.15) can be replaced by

$$\lim_{j\to+\infty} u^j(x^k) = w_f(x^k) \quad \text{for every} \quad k \in \mathbb{N}, \tag{5.16}$$

and each u^j satisfies

$$m_* \leq u^j \leq m^*. \tag{5.17}$$

Consider $u^j_{\xi,\rho}$ (see Definition 5.15) instead of u^j. Then the sequence $u^j_{\xi,\rho}$ still satisfies (5.16) and (5.17), and its elements still belong to $\sigma_f(\overline{\Omega})$.

Exercise 5.13: Prove the above statement about $u^j_{\xi,\rho}$.

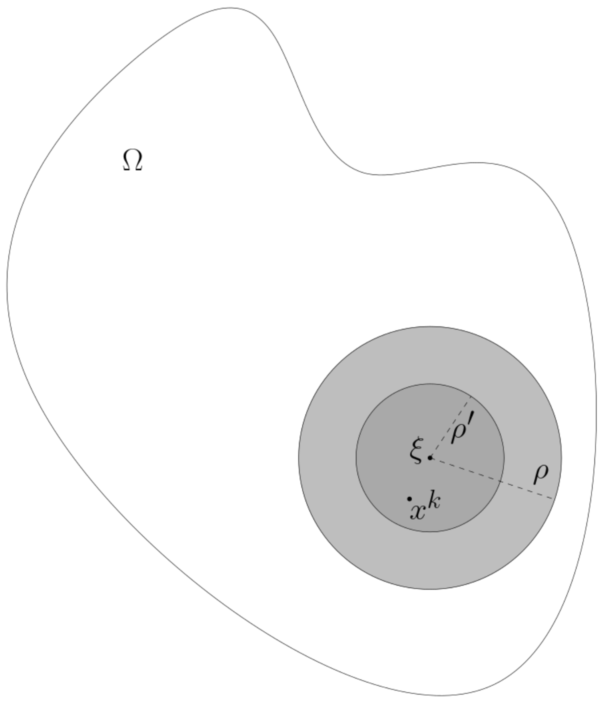


Figure 5.2: The construction in the proof of Proposition 5.22.

In addition, the sequence $u^j_{\xi,\rho}$ is harmonic in $B(\xi,\rho)$. From Proposition 5.14 (and using (5.17)) it follows that there exists a subsequence $u^{j_\ell}_{\xi,\rho}(x)$ converging to a harmonic function $W(x)$ for all $x \in \overline{B}(\xi,\rho')$. *The function W could depend upon the choice of points x^k and upon the choice of subsequence j_ℓ.*

Exercise 5.14: Prove the above statement about the convergence of a subsequence and its limit W being harmonic.

By (5.16) it follows that

$$w_f(x^k) = W(x^k) \quad \text{for all} \quad k. \tag{5.18}$$

Continuity of w_f. Let $x \in B(\xi,\rho')$. Choose the sequence x^k so that it converges to x (see Figure 5.3). Corresponding to this sequence, there exists a function W as discussed above. Assuming, without loss of generality, that $x^1 = x$, we have $W(x) = W(x^1)$. We know that W is harmonic in $B(\xi,\rho')$, so in particular it is continuous there, hence $\lim_{k\to\infty} W(x^k) = W(x)$. From (5.18) it follows that $\lim_{k\to\infty} w_f(x^k)$ also exists and is equal to $W(x) = W(x^1) = w_f(x^1)$. So w_f is continuous at x. Since x was arbitrary, $w_f \in C^0(B(\xi,\rho'))$.

Harmonicity of w_f. Now choose the sequence x^k to be dense in $B(\xi,\rho')$. Then w_f agrees with the corresponding harmonic function W on this dense set. Therefore, by continuity, $w_f = W$ everywhere in $B(\xi,\rho')$.

This shows that w_f is continuous and harmonic in a neighborhood of ξ. Since $\xi \in \Omega$ was arbitrary, we find that w_f is continuous and harmonic in Ω. $\square$

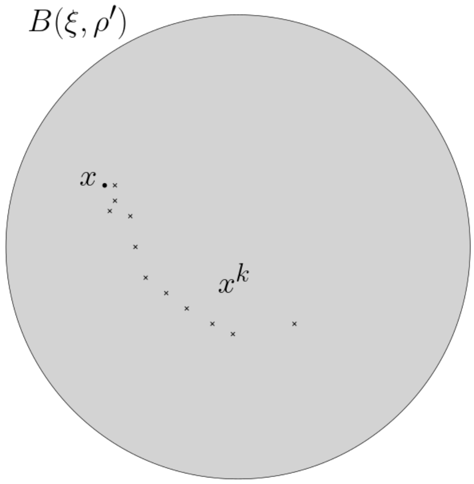


Figure 5.3: Continuity of w_f.

5.4.3 The Barrier Postulate

To prove that for each $\eta \in \partial\Omega$ we have that $\lim_{\substack{x \in \Omega \\ x \to \eta}} w_f(x) = f(\eta)$, we impose a new condition on the boundary: [5]

Barrier Postulate

Assume that for each $\eta \in \partial\Omega$ there exists a **barrier function**: a function $Q_\eta \in C^0(\overline{\Omega}) \cap \sigma(\Omega)$ such that

$$Q_\eta(\eta) = 0 \quad \text{and} \quad Q_\eta(x) < 0 \quad \text{for all} \quad x \in \partial\Omega \setminus \{\eta\}.$$

Proposition 5.23: For each $\eta \in \partial\Omega$,

$$\liminf_{\substack{x \in \Omega \\ x \to \eta}} w_f(x) \geq f(\eta). \tag{5.19}$$

Proof. Fix constants $\varepsilon, K > 0$ and consider the function

$$u(x) = f(\eta) - \varepsilon + KQ_\eta(x) \quad \text{for all} \quad x \in \overline{\Omega}.$$

Then $u \in C^0(\overline{\Omega}) \cap \sigma(\Omega)$. Moreover,

$$u(x) < f(\eta) - \varepsilon \quad \text{for any} \quad x \in \partial\Omega \setminus \{\eta\}$$

and

$$u(\eta) = f(\eta) - \varepsilon.$$

The continuity of f implies that there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x \in \partial\Omega$ with $|x - \eta| < \delta$, $|f(x) - f(\eta)| < \varepsilon$. It follows that

$$u(x) < f(x) \quad \text{for all} \quad x \in \partial\Omega,\ |x - \eta| < \delta. \tag{5.20}$$

[5]Over the years this condition has been weakened, and finding yet weaker conditions remains an active field of research. Here we stick with this classical condition.

Since the function Q_η is negative for all $x \in \partial\Omega \setminus \{\eta\}$, it is *uniformly* negative for all $x \in \partial\Omega \setminus \{\eta\}$ with $|x - \eta| \geq \delta$.

Exercise 5.15: Prove the above statement.

It follows that by choosing K (depending on ε) large enough, we can ensure that (5.20) holds also for $x \in \partial\Omega$ with $|x - \eta| \geq \delta$. This implies that $u \in \sigma_f(\overline{\Omega})$, and it follows (from the definition of w_f) that

$$u(x) \leq w_f(x) \quad \text{for all} \quad x \in \Omega.$$

Consequently

$$f(\eta) - \varepsilon = \lim_{x \to \eta} u(x) \leq \liminf_{\substack{x \in \Omega \\ x \to \eta}} w_f(x).$$

Taking the limit $\varepsilon \to 0$ completes the proof. □

We are now ready to prove that the limit of w_f at the boundary exists, thereby completing the task of solving (5.1):

Proposition 5.24: For each $\eta \in \partial\Omega$,

$$\lim_{\substack{x \in \Omega \\ x \to \eta}} w_f(x) = f(\eta). \tag{5.21}$$

Proof. We already know that $\liminf_{\substack{x \in \Omega \\ x \to \eta}} w_f(x) \geq f(\eta)$, so we just need to prove that $\limsup_{\substack{x \in \Omega \\ x \to \eta}} w_f(x) \leq f(\eta)$. We do this by applying all previous arguments to $-f$, as follows. Consider the function w_{-f}, defined as

$$w_{-f}(x) = \sup_{v \in \sigma_{-f}(\overline{\Omega})} v(x).$$

Using the fact that $\sup(-v) = -\inf v$, we denote $U = -v$ to get

$$-w_{-f}(x) = \inf U(x)$$

where the infimum is taken over all $-U \in C^0(\overline{\Omega}) \cap \sigma(\Omega)$ satisfying $-U \leq -f$ on $\partial\Omega$. Any such U and any $v \in \sigma_f(\overline{\Omega})$ satisfy $v - U \leq 0$ on $\partial\Omega$. Since both v and $-U$ are subharmonic, the Weak Maximum Principle (Theorem 5.4) implies that $v - U \leq 0$ in Ω. Considering that

$$w_f(x) = \sup v(x) \quad \text{and} \quad -w_{-f}(x) = \inf U(x)$$

where the infimum and supremum are taken over the sets specified above, the fact that $v - U \leq 0$ in Ω implies that

$$w_f(x) \leq -w_{-f}(x) \quad \text{for all} \quad x \in \Omega.$$

The result of Proposition 5.23 applied to $w_{-f}(x)$ leads to

$$\liminf_{\substack{x \in \Omega \\ x \to \eta}} w_{-f}(x) \geq -f(\eta) \quad \text{for all} \quad \eta \in \partial\Omega.$$

Combining the last two observations we conclude that

$$\limsup_{\substack{x \in \Omega \\ x \to \eta}} w_f(x) \leq \limsup_{\substack{x \in \Omega \\ x \to \eta}} \left(-w_{-f}(x)\right) = -\liminf_{\substack{x \in \Omega \\ x \to \eta}} w_{-f}(x) \leq f(\eta)$$

which concludes the proof. □

We have therefore just proven:

Theorem 5.25: Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain satisfying the barrier postulate. Let $f \in C^0(\partial\Omega)$. Then the problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \quad u = f & \text{on} \quad \partial\Omega. \end{cases}$$

has a (unique) solution in $C^0(\overline{\Omega}) \cap C^2(\Omega)$.

Exercise 5.16: Why do we not require that Ω be connected? (recall that each time we use the Maximum Principle we implicitly use the fact that the domain is connected)

Exercise 5.17: Prove that a strictly convex domain satisfies the barrier postulate. [6]

[6] Ω is strictly convex if through each $\eta \in \partial\Omega$ there passes a hyperplane that intersects $\overline{\Omega}$ only at η.