6 The Vlasov-Maxwell System: Conditional Global Existence

The proof of existence and uniqueness for the Vlasov-Poisson system relied heavily on the elliptic structure of Poisson's equation in order to obtain bounds for the momentum of the "fastest" particle, $P(t)$. These tools are not available to us here, and, indeed, an *a priori* bound on $P(t)$ is still an open problem.[31]

6.1 The Glassey-Strauss Theorem

We recall that the Vlasov-Maxwell system describes the evolution of a pdf $f(t,x,p) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ due to electromagnetic forces. It reads

$$\frac{\partial f}{\partial t}(t,x,p) + v \cdot \nabla_x f(t,x,p) + (\mathbf{E}_f(t,x) + v \times \mathbf{B}_f(t,x)) \cdot \nabla_p f(t,x,p) = 0, \tag{6.1}$$

$$\nabla \cdot \mathbf{E}_f = \rho_f, \quad \nabla \cdot \mathbf{B}_f = 0, \quad \nabla \times \mathbf{E}_f = -\frac{\partial \mathbf{B}_f}{\partial t}, \quad \nabla \times \mathbf{B}_f = \mathbf{j}_f + \frac{\partial \mathbf{E}_f}{\partial t}, \tag{6.2}$$

where

$$\rho_f(t,x) = \int_{\mathbb{R}^n} f(t,x,p)\,\mathrm{d}p = \text{particle density},$$

$$\mathbf{j}_f(t,x) = \int_{\mathbb{R}^n} f(t,x,p) v\,\mathrm{d}p = \text{current density},$$

and

$$\begin{aligned} &\text{classical case} \qquad v = p, \\ &\text{relativistic case} \qquad v = \frac{p}{\sqrt{1+|p|^2}}. \end{aligned}$$

The most famous existence and uniqueness results – due to Glassey and Strauss [Glassey1986] – is *conditional* in the sense that it requires momenta to be bounded in time though such a condition is not *a priori* known to hold. In this section we shall sketch the proof.

Theorem 6.1 (Conditional Existence of Classical Solutions (Relativistic Case)). *Let $f_0(x,p) \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f_0 \geq 0$, and let $\mathbf{E}(0,x) = \mathbf{E}_0(x)$ and $\mathbf{B}(0,x) = \mathbf{B}_0(x)$ be such that $\nabla \cdot \mathbf{E}_0 = \rho_{f_0}$ and $\nabla \cdot \mathbf{B}_0 = 0$. Assume that there exists a function $\beta(t)$ such that for all x, $f(t,x,p) = 0$ for $|p| > \beta(t)$. Then there exists a unique classical global solution $f(t,x,p)$ for the system* (6.1)-(6.2) *with $f(0,\cdot,\cdot) = f_0$ in the* relativistic case.

Remark 6.2. Note that the electromagnetic fields satisfy wave equations which require additional initial data (see Section 5). Hence the initial data $(\mathbf{E}_0(x), \mathbf{B}_0(x))$ must be complemented by another pair $(\mathbf{E}_1(x), \mathbf{B}_1(x))$. However these (dynamic) conditions are not reflected in the (static) problem at time $t = 0$.

6.2 A New Basis in Spacetime

Observe that the fields satisfy the wave equations

$$\begin{aligned} -\Box \mathbf{E}_f &= (\partial_t^2 - \Delta_x)\mathbf{E}_f = -\nabla_x \rho_f - \partial_t \mathbf{j}_f, \\ -\Box \mathbf{B}_f &= (\partial_t^2 - \Delta_x)\mathbf{B}_f = \nabla_x \times \mathbf{j}_f. \end{aligned}$$

[31] Such bounds exists in some special cases, for instance if assuming the initial data is "small" in some sense [Glassey1987a], or if the system possesses some symmetries [Glassey1990, Glassey1997].

with initial data $(\mathbf{E}_0, \mathbf{E}_1)$ and $(\mathbf{B}_0, \mathbf{B}_1)$, respectively. We know that there is no gain in regularity – that is, the regularity of the fields is *at most* the same regularity as of the right hand side. In the proof that shall follow below, we will employ an iterative scheme similar to the one used for solving Vlasov-Poisson: the vector field of the Nth iterate will be obtained from the fields of the $N-1$st iterate which, in turn, are solutions to wave equations. The right hand side of these equations contain moments of f in p differentiated in x. Hence, *a priori* we expect to have a loss in derivatives.

This loss of derivatives can be overcome if replacing the usual coordinates in spacetime by carefully chosen ones that respect the symmetries of the problem.

6.2.1 Representation of the Fields and Their Derivatives

It is convenient to work in coordinates that respect the symmetries of the Vlasov equation (free transport) and of Maxwell's equations (the light cone). That is, we wish to replace the partial derivatives ∂_t and ∂_{x_i} by suitably chosen directional derivatives. Fix a point $x \in \mathbb{R}^3$. A signal arriving at x at time t from a different point $y \in \mathbb{R}^3$ would have had to leave y at time $t - |x-y|$ (we have taken the speed of light to be 1). Hence we define:

$$\begin{aligned} S &:= \partial_t + v \cdot \nabla_x \\ T_i f &:= \partial_{y_i}[f(t - |x-y|, y, p)], \quad i = 1,2,3. \end{aligned}$$

Inverting, one has the representation:

$$\begin{aligned} \partial_t &= \frac{S - v \cdot T}{1 + v \cdot \omega} \\ \partial_{x_i} &= T_i + \frac{\omega_i}{1 + v \cdot \omega}(S - v \cdot T) = \frac{\omega_i S}{1 + v \cdot \omega} + \sum_{j=1}^{3} \left(\delta_{ij} - \frac{\omega_i v_j}{1 + v \cdot \omega} \right) T_j. \end{aligned}$$

where

$$\omega = \frac{y - x}{|y - x|}.$$

Representation of the Fields. We use the new coordinates to express the fields.

Proposition 6.3. *Under the hypotheses of Theorem 6.1 the fields admit the following representation:*

$$\mathbf{E}(t,x) = \mathbf{E}_0(t,x) + \mathbf{E}_T(t,x) + \mathbf{E}_S(t,x) \quad \textit{and} \quad \mathbf{B}(t,x) = \mathbf{B}_0(t,x) + \mathbf{B}_T(t,x) + \mathbf{B}_S(t,x)$$

where

$$\begin{aligned} \mathbf{E}_T^i(t,x) &= -\int_{|y-x| \le t} \int_{\mathbb{R}^3} \frac{(\omega_i + v_i)(1 - |v|^2)}{(1 + v \cdot \omega)^2} f(t - |y-x|, y, p) \, \mathrm{d}p \, \frac{\mathrm{d}y}{|y-x|^2}, \\ \mathbf{E}_S^i(t,x) &= -\int_{|y-x| \le t} \int_{\mathbb{R}^3} \frac{\omega_i + v_i}{1 + v \cdot \omega} (Sf)(t - |y-x|, y, p) \, \mathrm{d}p \, \frac{\mathrm{d}y}{|y-x|} \end{aligned}$$

and similar expressions for $\mathbf{B}$.

Proof. Let us show this for $\mathbf{E}$. We know that

$$-\Box \mathbf{E}^i = -\partial_{x_i} \rho - \partial_t \mathbf{j}^i = -\int_{\mathbb{R}^3} (\partial_{x_i} f + v \partial_t f) \, \mathrm{d}p.$$

We express the operator $\partial_{x_i} + v\partial_t$ appearing in the integrand on the right hand side as

$$\partial_{x_i} + v\partial_t = \frac{(\omega_i + v_i)S}{1 + v \cdot \omega} + \sum_{j=1}^{3} \left(\delta_{ij} - \frac{(\omega_i + v_i)v_j}{1 + v \cdot \omega} \right) T_j.$$

Now one proceeds by applying *Duhamel's principle* (Theorem 5.16) to the resulting inhomogeneous wave equation for $\mathbf{E}^i$ and integrating by parts (the full details can be found in [Glassey1996]). The proof for $\mathbf{B}$ is analogous. □

Representation of the Derivatives of the Fields. We have expressions analogous to the one obtained in Proposition 6.3:

Proposition 6.4. *Under the hypotheses of Theorem 6.1 the partial derivatives of the fields admit the following representation, for* $i, k = 1, 2, 3$*:*

$$\partial_k \mathbf{E}^i = (\partial_k \mathbf{E}^i)_0 + \int_{r \le t} \int_{\mathbb{R}^3} a(\omega, v) f \,\mathrm{d}p \,\frac{\mathrm{d}y}{r^3} + \int_{|\omega|=1} \int_{\mathbb{R}^3} d(\omega, v) f(t, x, p) \,\mathrm{d}\omega \,\mathrm{d}p$$
$$+ \int_{r \le t} \int_{\mathbb{R}^3} b(\omega, v) S f \,\mathrm{d}p \,\frac{\mathrm{d}y}{r^2} + \int_{r \le t} c(\omega, v) S^2 f \,\mathrm{d}p \,\frac{\mathrm{d}y}{r}$$

where f, Sf, S^f *without explicit arguments are evaluated at* $(t - |x - y|, y, p)$ *and* $r = |y - x|$. *The functions* a, b, c, d *are smooth except at* $1 + v \cdot \omega = 0$*, have algebraic singularities at such points, and* $\int_{|\omega|=1} a(\omega, v) \,\mathrm{d}\omega = 0$. *Therefore the first integral converges if* f *is sufficiently smooth. Similar expressions exist for the derivatives of* $\mathbf{B}$.

Proof. This is obtained from applying $\frac{\partial}{\partial x_k}$ to the expressions obtained in Proposition 6.3. For the long computation involved we refer to [Glassey1996]. For simplicity and future reference we shall write the expression for $\partial_k \mathbf{E}^i$ as:

$$\partial_k \mathbf{E}^i = \partial_k \mathbf{E}^i_0 + \partial_k \mathbf{E}^i_{TT} - \partial_k \mathbf{E}^i_{TS} + \partial_k \mathbf{E}^i_{ST} - \partial_k \mathbf{E}^i_{SS}$$

□

6.3 *A Priori* Estimates

6.3.1 A Uniform Bound for the Particle Density

Proposition 6.5. *The particle density satisfies the bound:*

$$\|f(t, \cdot, \cdot)\|_{C^1} \le c + c_T \int_0^t [1 + \|\mathbf{E}(\tau, \cdot)\|_{C^1} + \|\mathbf{B}(\tau, \cdot)\|_{C^1}] \, \|f(\tau, \cdot, \cdot)\|_{C^1} \,\mathrm{d}\tau \qquad (6.3)$$

for all $t \in [0, T]$.

Proof. Let $D \in \{\partial_{x_j}\}_{j=1}^3$ and denote $\mathbf{F} = \mathbf{E}_f(t, x) + v \times \mathbf{B}_f(t, x)$. Then

$$(\partial_t + v \cdot \nabla_x + \mathbf{F} \cdot \nabla_p)(Df) = -D\mathbf{F} \cdot \nabla_p f.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}s} Df(s, X(s; t, x, p), V(s; t, x, p)) = -D\mathbf{F} \cdot \nabla_p f(s, X(s; t, x, p), V(s; t, x, p))$$

which leads to the estimate

$$|Df(t, x, p)| \le |Df(0, X(0; t, x, p), V(0; t, x, p))|$$
$$+ \int_0^t |D\mathbf{F} \cdot \nabla_p f(s, X(s; t, x, p), V(s; t, x, p))| \,\mathrm{d}s.$$

A similar bound can be obtained for $D \in \{\partial_{p_j}\}_{j=1}^3$. The assertion follows from these two bounds. $\square$

6.3.2 A Uniform Bound for the Fields

Proposition 6.6. *The fields admit the uniform bound*

$$\|\mathbf{E}(t,\cdot)\|_{C^0} + \|\mathbf{B}(t,\cdot)\|_{C^0} \le c_T, \qquad \forall t \in [0,T]. \tag{6.4}$$

Proof. We omit this proof which is technical (though it is at the heart of the proof). Here, the assumption on the existence of a bound $\beta(t)$ of momenta is used crucially. $\square$

6.3.3 A Uniform Bound for the Gradients of the Fields

Proposition 6.7. *Let* $\log^* s = \begin{cases} s & s \le 1, \\ 1 + \ln s & s \ge 1. \end{cases}$ *Then the gradients of the fields admit the uniform bound*

$$\|\mathbf{E}(t,\cdot)\|_{C^1} + \|\mathbf{B}(t,\cdot)\|_{C^1} \le c_T \left[1 + \log^*\left(\sup_{\tau \le t} \|f(\tau,\cdot,\cdot)\|_{C^1}\right)\right], \qquad \forall t \in [0,T]. \tag{6.5}$$

Proof. We omit this proof which is technical (though it is at the heart of the proof). Here, the assumption on the existence of a bound $\beta(t)$ of momenta is used crucially. $\square$

6.4 Defining Approximate Solutions

The construction of approximate solutions follows the same ideas as in the proof for Vlasov-Poisson. However, now Poisson's (elliptic) equation is replaced by Maxwell's (hyperbolic) equations, and hence defining the vector field is more involved.

Set

$$f^0(t,x,p) = f_0(x,p), \quad \mathbf{E}^0(t,x) = \mathbf{E}_0(x), \quad \text{and} \quad \mathbf{B}^0(t,x) = \mathbf{B}_0(x),$$

and define

$$\rho^0(t,x) = \int_{\mathbb{R}^3} f^0(t,x,p)\,\mathrm{d}p \quad \text{and} \quad \mathbf{j}^0(t,x) = \int_{\mathbb{R}^3} f^0(t,x,p) v\,\mathrm{d}p.$$

Suppose that $(f^{N-1}, \mathbf{E}^{N-1}, \mathbf{B}^{N-1})$ have been defined and define f^N to be the solution to the linear transport equation

$$\begin{cases} \partial_t f^N(t,x,p) + v \cdot \nabla_x f^N(t,x,p) + \big(\mathbf{E}^{N-1}(t,x) + v \times \mathbf{B}^{N-1}(t,x)\big) \cdot \nabla_p f^N(t,x,p) = 0, \\ f^N(0,\cdot,\cdot) = f_0. \end{cases}$$

Hence one can write

$$f^N(t,x,p) = f_0(X^{N-1}(0;t,x,p), V^{N-1}(0;t,x,p)),$$
$$\rho^N(t,x) = \int_{\mathbb{R}^3} f^N(t,x,p)\,\mathrm{d}p \quad \text{and} \quad \mathbf{j}^N(t,x) = \int_{\mathbb{R}^3} f^N(t,x,p) v\,\mathrm{d}p,$$

and we can define $\mathbf{E}^N$ and $\mathbf{B}^N$ to be the solutions of

$$-\Box \mathbf{E}^N = (\partial_t^2 - \Delta_x)\mathbf{E}^N = -\nabla_x \rho^N - \partial_t \mathbf{j}^N,$$
$$-\Box \mathbf{B}^N = (\partial_t^2 - \Delta_x)\mathbf{B}^N = \nabla_x \times \mathbf{j}^N.$$

with initial data $(\mathbf{E}_0, \mathbf{E}_1)$ and $(\mathbf{B}_0, \mathbf{B}_1)$, respectively.

Goal: show that under the assumption that an upper bound $\beta(t)$ to momenta exists $\lim_{N\to\infty} f^N$ exists in $C^1([0,T)\times\mathbb{R}^6)$, $\lim_{N\to\infty}(\mathbf{E}^N,\mathbf{B}^N)$ exists in $C^1([0,T)\times\mathbb{R}^3)$, that the limits satisfy the relativistic Vlasov-Maxwell system (uniquely).

6.4.1 The Iterates are Well-Defined

Lemma 6.8. *If $f^N \in C^2([0,T)\times\mathbb{R}^6)$ then $\mathbf{E}^N, \mathbf{B}^N \in C^2([0,T)\times\mathbb{R}^3)$.*

Proof. Recall that $\mathbf{E}^N$ and $\mathbf{B}^N$ satisfy the wave equations

$$-\Box\mathbf{E}^N = -\nabla_x\rho^N - \partial_t\mathbf{j}^N,$$
$$-\Box\mathbf{B}^N = \nabla_x\times\mathbf{j}^N.$$

If $f^N \in C^2$ then the right hand sides of these equations are in C^1 and hence so are the fields. To show that they are in fact C^2 we need to employ the representation results and proceed by induction. Recall that

$$\mathbf{E}^N(t,x) = \mathbf{E}_0(t,x)+\mathbf{E}_T^N(t,x)+\mathbf{E}_S^N(t,x) \quad \text{and} \quad \mathbf{B}^N(t,x) = \mathbf{B}_0(t,x)+\mathbf{B}_T^N(t,x)+\mathbf{B}_S^N(t,x).$$

The data terms $\mathbf{E}_0(t,x)$ and $\mathbf{B}_0(t,x)$ are C^2, so we only need to analyse the other terms. Take for instance the expression

$$\mathbf{E}_S^N(t,x) = -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{\omega+v}{1+v\cdot\omega}(Sf^N)(t-|y-x|,y,p)\,\mathrm{d}p\,\frac{\mathrm{d}y}{|y-x|}.$$

Notice that

$$Sf^N = -\nabla_p\cdot\left[(\mathbf{E}^{N-1}+v\times\mathbf{B}^{N-1})f^N\right]$$

which allows us to integrate by parts in p and use the induction hypothesis that $\mathbf{E}^{N-1}$ and $\mathbf{B}^{N-1}$ are C^2. A similar argument can be employed for $\mathbf{E}_T^N(t,x)$. Hence $\mathbf{E}^N$ is C^2 and the same holds for $\mathbf{B}^N$. $\Box$

6.4.2 The Limit $\lim_{N\to\infty}(f^N,\mathbf{E}^N,\mathbf{B}^N)$ and its Properties

From the bounds on the fields and their gradients, (6.4) and (6.5) respectively, uniform bounds for the iterates follow. Similarly, we obtain uniform bounds for the particle density. Together with compactness, this shows that the sequences admit limits. However some of the elements of this proof are also lengthy and omitted here.

6.5 Uniqueness

For uniqueness we only require the expressions obtained in Proposition 6.3. This is also omitted here.