4 The Vlasov-Poisson System: Global Existence and Uniqueness

In this section we plan to show that the local existence result – Theorem 3.2 – can in fact be extended indefinitely, that is the maximal time of existence is $T = +\infty$. In the proof of the local result, we saw that the fields and their derivatives were all bounded by powers of the crucial quantity $P(t)$ – the momentum of the "fastest" particle at time t. Hence, as long as this quantity can be controlled, the solution can be continued. Previously, the optimal estimate on $P(t)$ we could obtain was related to the maximal solution of

$$P(t) = P_0 + C(f^0) \int_0^t P^2(s)\,\mathrm{d}s,$$

that is:

$$P(t) = \frac{P_0}{1 - P_0 C(f^0) t} \text{ and } \delta = \frac{1}{P_0 C(f^0)}.$$

Improving this bound shall require some more *a priori* estimates.

4.1 *A Priori* Estimates

In Section 3.2 we obtained some preliminary basic *a priori* estimates for the Vlasov-Poisson system

$$\frac{\partial f}{\partial t}(t,x,p) + p\cdot\nabla_x f(t,x,p) - \gamma\nabla\phi_f(t,x)\cdot\nabla_p f(t,x,p) = 0, \tag{4.1}$$

$$-\Delta\phi_f(t,x) = \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \qquad \lim_{|x|\to\infty}\phi_f(t,x) = 0. \tag{4.2}$$

with initial data

$$f(0,\cdot,\cdot) = f_0 \in C_0^1(\mathbb{R}^6),$$

where $\gamma = \pm 1$ differentiates between repulsive ($+1$) and attractive (-1) dynamics. Let us recall these basic estimates. First, we identified the conservation of all L^p norms, $p \in [1,\infty]$:

$$\|f(t,\cdot,\cdot)\|_p = \|f_0\|_p$$

as long as the solution exists. Then we recalled estimates related to the solution of Poisson's equation $-\Delta\phi = \rho$ in $\mathbb{R}^3$: For any $p \in [1,3)$

$$\|\nabla\phi_\rho\|_\infty \le c_p \|\rho\|_p^{p/3}\|\rho\|_\infty^{1-p/3} \qquad (c_p \text{ only depends on } p),$$

and for any $p \in [1,3)$, $R > 0$ and $d \in (0,R]$, $\exists c > 0$ independent of ρ, R, d, s.t.

$$\|D^2\phi_\rho\|_\infty \le c\left(\frac{\|\rho\|_1}{R^3} + d\|\nabla\rho\|_\infty + (1+\ln(R/d))\|\rho\|_\infty\right),$$

$$\|D^2\phi_\rho\|_\infty \le c(1+\|\rho\|_\infty)(1+\ln_+\|\nabla\rho\|_\infty) + c\|\rho\|_1.$$

These estimates are insufficient if one wants to bound momenta. For that we need to use the conservative nature of the Vlasov-Poisson system. More precisely, in what follows not only will we show that one can define an energy and that it is conserved, we will also show that we can bound the *kinetic* energy.

Lemma 4.1 (Continuity Equation). *Let $T > 0$ be the time of local existence of a solution f to the Vlasov-Poisson system (as given in Theorem 3.2). Let $t \in [0, T)$ and define the (vector-valued)* ***flux****:*

$$j_f(t,x) = \int_{\mathbb{R}^3} p f(t,x,p)\,\mathrm{d}p.$$

Then the following (continuity) equation holds for every $x \in \mathbb{R}^3$:

$$\partial_t \rho_f + \nabla_x \cdot j_f = 0. \tag{4.3}$$

Proof. Integrate the Vlasov equation in p and eliminate the last term due to the divergence theorem. □

Definition 4.2 (Energy). *For a solution $f(t,x,p)$ of the Vlasov-Poisson system we define its* ***kinetic energy***

$$\mathcal{E}_{kin}(t) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{2}|p|^2 f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p,$$

its ***potential energy***

$$\mathcal{E}_{pot}(t) := \frac{\gamma}{2}\int_{\mathbb{R}^3} |\nabla\phi_f(t,x)|^2\,\mathrm{d}x = \frac{\gamma}{2}\int_{\mathbb{R}^3} \rho_f(t,x)\phi_f(t,x)\,\mathrm{d}x$$

and ***total energy***

$$\mathcal{E}(t) := \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t).$$

Remark 4.3 (Attractive vs Repulsive Dynamics). Notice that the kinetic energy is a positive. However, the potential energy is only positive in the repulsive (plasma) regime; in the attractive (galactic) regime, the *potential energy is negative*. Hence in the **repulsive case**

$$\mathcal{E}(t) = \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t), \qquad \mathcal{E}_{kin}(t), \mathcal{E}_{pot}(t) \geq 0,$$

since $\gamma = +1$, and in the **attractive case**

$$\mathcal{E}(t) = \mathcal{E}_{kin}(t) - |\mathcal{E}_{pot}(t)|, \qquad \mathcal{E}_{kin}(t) \geq 0, \mathcal{E}_{pot}(t) \leq 0,$$

since $\gamma = -1$. Assuming for the moment that the total energy is conserved (we will show this later), we immediately obtain a uniform bound for the potential and kinetic energies in the *repulsive* case. However, in the *attractive* case there's no *a priori* reason why both energies cannot blow up, while their sum remains constant. We will show that this does not happen.

Proposition 4.4 (Conservation of Total Energy). *The total energy $\mathcal{E}(t)$ is conserved.*

Proof. Multiply the Vlasov equation by $\frac{|p|^2}{2}$ and integrate in (x,p) to obtain

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^6} \frac{|p|^2}{2}\partial_t f(t,x,p) + \iint_{\mathbb{R}^6} \frac{|p|^2}{2} p\cdot\nabla_x f(t,x,p) - \gamma\iint_{\mathbb{R}^6} \frac{|p|^2}{2}\nabla_x\phi_f(t,x)\cdot\nabla_p f(t,x,p) \\
&= \dot{\mathcal{E}}_{kin}(t) + \iint_{\mathbb{R}^6} \nabla_x\cdot\frac{|p|^2}{2} p f(t,x,p) + \gamma\iint_{\mathbb{R}^6} \nabla_x\phi_f(t,x)\cdot p f(t,x,p) \\
&= \dot{\mathcal{E}}_{kin}(t) + 0 - \gamma\int_{\mathbb{R}^3} \phi_f(t,x)\nabla_x\cdot j_f(t,x) \\
&= \dot{\mathcal{E}}_{kin}(t) + \gamma\int_{\mathbb{R}^3} \phi_f(t,x)\partial_t\rho_f(t,x) \\
&= \dot{\mathcal{E}}_{kin}(t) + \frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \phi_f(t,x)\rho_f(t,x) = \dot{\mathcal{E}}_{kin}(t) + \dot{\mathcal{E}}_{pot}(t) = \dot{\mathcal{E}}(t).
\end{aligned}$$

We used the continuity equation going from the second to the third line (exercise: verify the transition from the fourth to the fifth line) □

Proposition 4.5 (Bounds on the Energy). *Let $f(t,x,p)$ be a classical solution of Vlasov-Poisson on $[0,T)$. Then for all $t \in [0,T)$:*

$$\mathcal{E}_{kin}(t), |\mathcal{E}_{pot}(t)|, \|\rho_f(t,\cdot)\|_{L^{5/3}(\mathbb{R}^3)} \leq C$$

where $C = C(\|f_0\|_\infty, \|f_0\|_1, \mathcal{E}_{kin}(0))$.

Proving Proposition 4.5 will require the following lemma (which we prove later):

Lemma 4.6 (Moment Estimates). *Let $g = g(x,p) : \mathbb{R}^6 \to \mathbb{R}_+$ be measurable and define*

$$m_k(g)(x) := \int_{\mathbb{R}^3} |p|^k g(x,p)\,\mathrm{d}p$$
$$M_k(g) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} |p|^k g(x,p)\,\mathrm{d}p\,\mathrm{d}x$$

Let $p, p^* \in [1,\infty]$ satisfy $\frac{1}{p} + \frac{1}{p^*} = 1$, let $0 \leq k' \leq k < \infty$ and define

$$r = \frac{k + 3/p^*}{k' + 3/p^* + (k-k')/p}.$$

Then

$$\|m_{k'}(g)\|_r \leq c\|g\|_p^{\frac{k-k'}{k+3/p^*}} M_k(g)^{\frac{k'+3/p^*}{k+3/p^*}}$$

whenever the above quantities are finite, and $c = c(k,k',p) > 0$.

Proof of Proposition 4.5. Due to the conservation of total energy, the bounds on $\mathcal{E}_{kin}(t)$ and $\mathcal{E}_{pot}(t)$ in the repulsive case ($\gamma = +1$) are trivial. In the attractive case we use the Hardy-Littlewood-Sobolev inequality and Lemma 4.6 with $k = 2, k' = 0, p = 9/7, r = 6/5$ to obtain:

$$\begin{aligned}
|\mathcal{E}_{pot}(t)| &= \frac{1}{2}\int_{\mathbb{R}^3} \rho_f(t,x)\phi_f(t,x)\,\mathrm{d}x \\
&= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_f(t,x)\rho_f(t,y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq c\|\rho_f(t,\cdot)\|_{6/5}^2 \\
&\leq c\|f(t,\cdot,\cdot)\|_{9/7}^{3/2}\mathcal{E}_{kin}^{1/2}(t).
\end{aligned}$$

Now we use the conservation of energy to bound $\mathcal{E}_{kin}(t)$:

$$\begin{aligned}
\text{const} = \mathcal{E}(t) &= \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t) \\
&\geq \mathcal{E}_{kin}(t) - c\|f(t,\cdot,\cdot)\|_{9/7}^{3/2}\mathcal{E}_{kin}^{1/2}(t)
\end{aligned}$$

and since $c\|f(t,\cdot,\cdot)\|_{9/7}^{3/2}$ is some constant, the desired bound is achieved. For the bound on ρ, we use Lemma 4.6 with $k = 2, k' = 0, p = \infty, r = 5/3$ to obtain:

$$\|\rho_f(t,\cdot)\|_{5/3} \leq c\|f(t,\cdot,\cdot)\|_\infty^{2/5}\mathcal{E}_{kin}^{3/5}(t).$$

□

Proof of Lemma 4.6. This will be included in these notes in the future. For now, this is left as an exercise (you can also find the proof in textbooks). □

4.2 Remarks on Global Existence

Let us see if we can improve the Gronwall inequality stemming from the equation

$$P(t) = P(0) + C(f^0) \int_0^t P^2(s)\,\mathrm{d}s.$$

We know that $\|\nabla\phi_f\|_\infty \le c_p \|\rho_f\|_p^{p/3} \|\rho_f\|_\infty^{1-p/3}$, and so with $p = 5/3$ we obtain

$$\|\nabla\phi_f\|_\infty \le c \underbrace{\|\rho_f\|_{5/3}^{5/9}}_{\le \text{const}} \underbrace{\|\rho_f\|_\infty^{4/9}}_{\le (P^3(t))^{4/9}} \le c P^{4/3}(t).$$

Hence we obtain the improved bound

$$P(t) = P(0) + \int_0^t \|\nabla\phi_f(s,\cdot)\|_\infty \,\mathrm{d}s \le P(0) + c\int_0^t P^{4/3}(s)\,\mathrm{d}s.$$

This is better, but not good enough: the solution to this equation still has finite-time blowup. The exponent 4/3 comes from the estimate $\|\nabla\phi_f\|_\infty \le c_p \|\rho_f\|_p^{p/3} \|\rho_f\|_\infty^{1-p/3}$, and to lower it to an exponent that is ≤ 1 we would need the term $\|\rho_f\|_\infty$ to appear with a power no more than 1/3, hence $p = 2$. To summarise, using Lemma 4.6 we have

$$\begin{aligned}
\|\nabla\phi_f\|_\infty &\le c\|\rho_f\|_2^{2/3}\|\rho_f\|_\infty^{1/3} \\
&\le c\|\rho_f\|_2^{2/3} P(t) \\
&\le c\|f\|_\infty^{1/2} \underbrace{\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |p|^3 f(t,x,p)\,\mathrm{d}p\,\mathrm{d}x\right)^{1/2}}_{M_3^{1/2}(f)} P(t)
\end{aligned}$$

which leads to:

Proposition 4.7. *The breakdown criterion of Theorem 3.2 can be modified: as long as $\|\rho_f\|_2$ or $M_3(f)$ remain bounded, the solution does not blowup.*

4.3 Proof of Global Existence and Uniqueness

As of the typing of these notes there are two main approaches for proving global existence and uniqueness of solutions. The first, which we shall pursue, involves a detailed analysis of the trajectory of one particle thereby obtaining *a priori* bounds for the maximal momentum $P(t)$. This approach is due to Pfaffelmoser [Pfaffelmoser1992], later improved somewhat by Schaeffer [Schaeffer1991] (Pfaffelmoser's paper took a long time to publish, and was therefore published after Schaeffer's paper) . The other approach, due to Lions and Perthame [Lions1991] follows the idea of Proposition 4.7 by obtaining *a priori* estimates for higher moments of f. The main difference between these two approaches is in that the second approach does not require the initial datum f^0 to be compactly supported, while the first does. This makes the second approach somewhat more physically relevant.

As we have already seen (see Proposition 3.7 for instance) momentum growth is due to the field $\mathbf{E}_f = -\nabla\phi_f$ which is given by the relation

$$\nabla\phi_f(t,x) = -\int \frac{x-y}{|x-y|^3}\rho_f(t,y)\,\mathrm{d}y = -\iint \frac{x-y}{|x-y|^3} f(t,y,p)\,\mathrm{d}p\,\mathrm{d}y.$$

This leads to the estimate

$$|\nabla\phi_f(t,x)| \le \iint \frac{f(t,y,p)}{|x-y|^2}\,\mathrm{d}p\,\mathrm{d}y.$$

As in Section 3 we denote $T > 0$ to be the maximal time of local existence and uniqueness of the Vlasov-Poisson system, and for $s, t \in [0, T)$ the characteristics of the Vlasov-Poisson system $(X(s;t,x,p), V(s;t,x,p))$ are solutions to:

$$\dot{X}(s;t,x,p) = V(s;t,x,p)$$
$$\dot{V}(s;t,x,p) = -\gamma \nabla \phi_f(t, X(s;t,x,p))$$

with initial conditions

$$X(t;t,x,p) = x \qquad V(t;t,x,p) = p$$

so that

$$f(t,x,p) = f_0(X(0;t,x,p), V(0;t,x,p)).$$

We now follow the approach of Pfaffelmoser, and *fix a characteristic* $(\widetilde{X}(s), \widetilde{V}(s))$ corresponding to one particle (we don't care here about dependence upon the other parameters) with $(\widetilde{X}(0), \widetilde{V}(0)) \in \operatorname{supp} f_0$. We want to study the increase in momentum of this particle, satisfying the simple estimate

$$|\widetilde{V}(t) - \widetilde{V}(t-\Delta)| \leq \int_{t-\Delta}^{t} |\nabla \phi_f(s, \widetilde{X}(s))| \,\mathrm{d}s \leq \int_{t-\Delta}^{t} \iint \frac{f(s,y,w)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}w \,\mathrm{d}y \,\mathrm{d}s. \tag{4.4}$$

This shall be the main task in the proof of the following theorem:

Theorem 4.8 (Global Existence of Classical Solutions). *Let $f_0(x,p) \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f_0 \geq 0$ be given. Then there exists a unique classical global solution $f(t,x,p)$ for the system* (4.1)-(4.2) *with $f(0,\cdot,\cdot) = f_0$.*

Proof. We first recall (see Theorem 2.4) that the mapping $(X(s;t,\cdot,\cdot), V(s;t,\cdot,\cdot)) : \mathbb{R}^6 \to \mathbb{R}^6$ is orientation and measure preserving. Hence letting

$$y = X(s;t,x,p), \qquad w = V(s;t,x,p),$$
$$x = X(t;s,y,w), \qquad p = V(t;s,y,w),$$

and using the fact that f is constant along characteristics, the estimate (4.4) can be rewritten as

$$|\widetilde{V}(t) - \widetilde{V}(t-\Delta)| \leq \int_{t-\Delta}^{t} \iint \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p \,\mathrm{d}x \,\mathrm{d}s. \tag{4.5}$$

The goal in this proof will be to estimate this integral. We shall break up the domain of integration into three parts, *the good*, *the bad* and *the ugly*. Fix some parameters (to be made precise later) $q, r > 0$ and define:

$$M_g := \left\{ (s,x,p) \in [t-\Delta, t] \times \mathbb{R}^6 \;:\; |p| \leq q \,\vee\, |p - \widetilde{V}(t)| \leq q \right\}$$

$$M_b := \left\{ (s,x,p) \in [t-\Delta, t] \times \mathbb{R}^6 \;:\; |p| > q \,\wedge\, |p - \widetilde{V}(t)| > q \,\wedge \right.$$
$$\left. \wedge \left[|X(s;t,x,p) - \widetilde{X}(s)| \leq r|p|^{-3} \,\vee\, |X(s;t,x,p) - \widetilde{X}(s)| \leq r|p - \widetilde{V}(t)|^{-3} \right] \right\}$$

$$M_u := \left\{ (s,x,p) \in [t-\Delta, t] \times \mathbb{R}^6 \;:\; |p| > q \,\wedge\, |p - \widetilde{V}(t)| > q \,\wedge \right.$$
$$\left. \wedge\, |X(s;t,x,p) - \widetilde{X}(s)| > r|p|^{-3} \,\wedge\, |X(s;t,x,p) - \widetilde{X}(s)| > r|p - \widetilde{V}(t)|^{-3} \right\}.$$

The *good set* is the set on which either momenta are small, or momenta relative to the chosen particle are small.

The *bad set* is the set on which both the momenta and relative momenta are large, but the positions are close to the chosen particle.

The *ugly set* is the set on which momenta and relative momenta are large, and particles are far away from the chosen particle.

To proceed, we first redefine $P(t)$ (the maximal momentum at time t, originally defined in (3.9)) so that it is a monotonically increasing function:

$$P(t) := \sup_{\substack{(x,p)\in \operatorname{supp} f(s,\cdot,\cdot)\\ s\in[0,t]}} |p|. \tag{4.6}$$

Choice of Δ. We choose Δ to be sufficiently small so that momenta don't change too much between $t-\Delta$ and t. Recalling the estimate $\|\nabla\phi_f(t,\cdot)\|_\infty \le cP^{4/3}(t)$, we define

$$\Delta := \min\left\{t, \frac{q}{4cP^{4/3}(t)}\right\} \tag{4.7}$$

so that for all $s\in[t-\Delta,t]$, $(x,p)\in\mathbb{R}^6$,

$$|V(s;t,x,p)-p| \le \Delta c P^{4/3}(t) \le \frac{1}{4}q. \tag{4.8}$$

The Good Set. We will show that

$$\iiint_{M_g} \frac{f(t,x,p)}{|\widetilde{X}(s)-X(s;t,x,p)|^2}\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le cq^{4/3}\Delta. \tag{4.9}$$

The Bad Set. We will show that

$$\iiint_{M_b} \frac{f(t,x,p)}{|\widetilde{X}(s)-X(s;t,x,p)|^2}\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le cr\ln\left(\frac{4P(t)}{q}\right)\Delta. \tag{4.10}$$

The Ugly Set. We will show that

$$\iiint_{M_u} \frac{f(t,x,p)}{|\widetilde{X}(s)-X(s;t,x,p)|^2}\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le \frac{c}{r}. \tag{4.11}$$

Estimating the Integral. Before proceeding to prove (4.9),(4.10) and (4.11) we first add these estimates up to see what we get. They give:

$$\begin{aligned}
|\widetilde{V}(t)-\widetilde{V}(t-\Delta)| &\le c\left(q^{4/3} + r\ln\left(\frac{4P(t)}{q}\right) + \frac{1}{r\Delta}\right)\Delta\\
&\le c\left(q^{4/3} + r\ln\left(\frac{4P(t)}{q}\right) + \frac{1}{r}\max\left\{\frac{1}{t}, \frac{4cP^{4/3}(t)}{q}\right\}\right)\Delta.
\end{aligned}$$

Now we want to optimise the choice of q and r. Without loss of generality, there exists t for which $P(t)>1$ (otherwise we replace $P(t)$ by $P(t)+1$). Recall also that $P(t)$ is now defined to be monotonically increasing. Hence, defining

$$q = P^{4/11}(t) \qquad r = P^{16/33}(t),$$

we have that $q\le P(t)$. Moreover, if the maximal time of existence $T<\infty$, then we know that $P(t)\uparrow\infty$ as $T\uparrow\infty$. Hence there exists $T^*\in(0,T)$ such that $\frac{1}{t}\le\frac{4cP^{4/3}(t)}{q} = 4cP^{32/33}(t)$ for all $t\in[T^*,T)$. Therefore, for all $t\in[T^*,T)$

$$|\widetilde{V}(t)-\widetilde{V}(t-\Delta)| \le cP^{16/33}(t)\ln P(t)\Delta$$

so that for any $\varepsilon > 0$ there exists some $c = c(\varepsilon) > 0$ such that

$$|\widetilde{V}(t) - \widetilde{V}(t-\Delta)| \le cP^{16/33+\varepsilon}(t)\Delta. \tag{4.12}$$

Now, notice that $\Delta(t)$ is monotonically decreasing on (T^*, T). Fix some $t \in (T^*, T)$, and define

$$\begin{aligned} t_0 &= t \\ t_{i+1} &= t_i - \Delta(t_i). \end{aligned}$$

Since $\Delta(t)$ is monotonically decreasing, there exists some k such that

$$t_k < T^* \le t_{k-1} < t_{k-2} < \cdots < t_1 < t_0 = t.$$

Hence from (4.12) we get

$$\begin{aligned} |\widetilde{V}(t) - \widetilde{V}(t_k)| &\le \sum_{i=1}^{k} |\widetilde{V}(t_{i-1}) - \widetilde{V}(t_i)| \\ &\le cP^{16/33+\varepsilon}(t) \sum_{i=1}^{k} (t_{i-1} - t_i) \\ &\le cP^{16/33+\varepsilon}(t)t. \end{aligned}$$

However this implies (due to the very definition of $P(t)$) that $P(t) \le P(t_k) + cP^{16/33+\varepsilon}(t)t$ so that for any $\delta > 0$ there exists some $c = c(\delta) > 0$ such that

$$P(t) \le c(1+t)^{33/17+\delta}, \qquad \forall t \in [0, T)$$

which implies that $T = \infty$ (see Theorem 3.2) and finishes the proof. Hence we are only left with verifying the estimates (4.9),(4.10) and (4.11).

The Good: Proof of (4.9)**.** We want to show that on the *good set*, the following holds:

$$\iiint_{M_g} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \le cq^{4/3}\Delta.$$

Recall (4.8): $|V(s;t,x,p) - p| \le \frac{1}{4}q$. Then if $(s,x,p) \in M_g$, in the original coordinates (s,y,w) the following holds:

$$|w| < 2q \,\vee\, |w - \widetilde{V}(s)| < 2q.$$

Hence we get

$$\begin{aligned} &\iiint_{M_g} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le \int_{t-\Delta}^{t} \int_{\mathbb{R}^3} \int_{|w|<2q \,\vee\, |w-\widetilde{V}(s)|<2q} \frac{f(s,y,w)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}s \\ &\quad= \int_{t-\Delta}^{t} \int_{\mathbb{R}^3} \frac{\tilde{\rho}(s,y)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}y\,\mathrm{d}s \qquad \left[\text{where } \tilde{\rho}(s,y) := \int_{|w|<2q \,\vee\, |w-\widetilde{V}(s)|<2q} f(s,y,w)\,\mathrm{d}w\right] \\ &\quad\le c\int_{t-\Delta}^{t} \underbrace{\|\tilde{\rho}(s,\cdot)\|_{5/3}^{5/9}}_{\le \text{const}} \underbrace{\|\tilde{\rho}(s,\cdot)\|_{\infty}^{4/9}}_{\le (cq^3)^{4/9}} \,\mathrm{d}s \qquad \text{[by Propositions 3.7 and 4.5]} \\ &\quad\le cq^{4/3}\Delta. \end{aligned}$$

The Bad: Proof of (4.10)**.** We want to show that on the *bad set*, the following holds:

$$\iiint_{M_b} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \leq cr \ln\left(\frac{4P(t)}{q}\right)\Delta.$$

Recall again (4.8): $|V(s;t,x,p) - p| \leq \frac{1}{4}q$. If $(s,x,p) \in M_b$, in the original coordinates (s,y,w) the following holds:

$$\frac{1}{2}q < |w| < 2|p| \wedge \frac{1}{2}q < |w - \widetilde{V}(s)| < 2|p - \widetilde{V}(t)| \wedge$$
$$\wedge \left[|y - \widetilde{X}(s)| \leq 8r|w|^{-3} \vee |y - \widetilde{X}(s)| \leq 8r|w - \widetilde{V}(s)|^{-3}\right].$$

Now, we also use the fact that for $s \in [0,t]$ and $w \in \operatorname{supp} f(s,y,\cdot)$ it holds that $|w| \leq P(t)$, $|w - \widetilde{V}(s)| \leq 2P(t)$. Using also the conservation of L^∞ norms, we estimate

$$\begin{aligned}
&\iiint_{M_b} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \int_{t-\Delta}^{t} \int_{\frac{1}{2}q<|w|<2P(t)} \int_{|y-\widetilde{X}(s)|<8r|w|^{-3}} \frac{f(s,y,w)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \\
&\quad + \int_{t-\Delta}^{t} \int_{\frac{1}{2}q<|w-\widetilde{V}(s)|<2P(t)} \int_{|y-\widetilde{X}(s)|<8r|w-\widetilde{V}(s)|^{-3}} \frac{f(s,y,w)}{|\widetilde{X}(s) - y|^2} \,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \\
&\leq c \int_{t-\Delta}^{t} \int_{\frac{1}{2}q<|w|<2P(t)} 4\pi \cdot 8r|w|^{-3} \,\mathrm{d}w\,\mathrm{d}s \\
&\quad + c \int_{t-\Delta}^{t} \int_{\frac{1}{2}q<|w-\widetilde{V}(s)|<2P(t)} 4\pi \cdot 8r|w - \widetilde{V}(s)|^{-3} \,\mathrm{d}w\,\mathrm{d}s \\
&\leq cr \ln\left(\frac{4P(t)}{q}\right)\Delta
\end{aligned}$$

where we have integrated in y and in w by changing to spherical coordinates.

The Ugly: Proof of (4.11)**.** In estimating the integral over the ugly set we shall, for the first time, use some smoothing properties of the time integral, rather than simply estimate it by Δ. Recall, that we want to show that

$$\iiint_{M_u} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s \leq \frac{c}{r}$$

where M_u is defined as

$$M_u := \Big\{(s,x,p) \in [t-\Delta,t] \times \mathbb{R}^6 \;:\; |p| > q \wedge |p - \widetilde{V}(t)| > q \wedge$$
$$\wedge\, |X(s;t,x,p) - \widetilde{X}(s)| > r|p|^{-3} \wedge |X(s;t,x,p) - \widetilde{X}(s)| > r|p - \widetilde{V}(t)|^{-3}\Big\}.$$

First, we want a lower bound for the distance between the particles, which we denote, for $s \in [t-\Delta,t]$,

$$D(s) := X(s;t,x,p) - \widetilde{X}(s).$$

Note that $D : [t-\Delta,t] \to \mathbb{R}^3$. We claim that for all $s \in [t-\Delta,t]$ and $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $|p - \widetilde{V}(t)| > q$ the following lower bound holds:

$$|D(s)| \geq \frac{1}{4}|p - \widetilde{V}(t)||s - s_0|. \tag{4.13}$$

We prove this by comparing to a linear approximation, defined as follows. Let $s_0 \in [t-\Delta, t]$ be such that $|D(s)|$ attains a minimum there and let

$$\overline{D}(s) := D(s_0) + \dot{D}(s_0)(s - s_0)$$

be the tangent (i.e. linear approximation) to $D(s)$ at s_0. Hence D and $\overline{D}$ and their first derivatives agree at s_0. As for the second derivative, we have

$$|\ddot{D}(s) - \ddot{\overline{D}}(s)| = |\dot{V}(s; t, x, p) - \dot{\widetilde{V}}(s)| \leq 2\|\nabla\phi_f(s, \cdot)\|_\infty \leq cP^{4/3}(t).$$

Therefore, a simple Taylor expansion gives

$$\begin{aligned} |D(s) - \overline{D}(s)| &\leq cP^{4/3}(t)(s - s_0)^2 \\ &\leq cP^{4/3}(t)\Delta|s - s_0| \\ &\leq \frac{1}{4}q|s - s_0| \\ &< \frac{1}{4}|p - \widetilde{V}(t)||s - s_0|. \end{aligned} \tag{4.14}$$

Next, let us show that

$$|\overline{D}(s)|^2 \geq \frac{1}{4}|p - \widetilde{V}(t)|^2|s - s_0|^2. \tag{4.15}$$

Indeed, observe that

$$\begin{aligned} |p - \widetilde{V}(t)| &\leq |p - V(s_0; t, x, p)| + |\widetilde{V}(s_0) - \widetilde{V}(t)| + |V(s_0; t, x, p) - \widetilde{V}(s_0)| \\ &\leq \frac{1}{2}q + |V(s_0; t, x, p) - \widetilde{V}(s_0)| \end{aligned}$$

so that

$$|\dot{D}(s_0)| = |V(s_0; t, x, p) - \widetilde{V}(s_0)| \geq |p - \widetilde{V}(t)| - \frac{1}{2}q > \frac{1}{2}|p - \widetilde{V}(t)|.$$

By the definition of s_0 we have that $(s - s_0)D(s_0) \cdot \dot{D}(s_0) \geq 0$ for all $s \in [t - \Delta, t]$, which is enough to prove (4.15). Combining (4.14) and (4.15) we have (4.13).

Now define functions $\sigma_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, as

$$\sigma_1(\xi) := \begin{cases} \xi^{-2} & \xi > r|p|^{-3} \\ (r|p|^{-3})^{-2} & \xi \leq r|p|^{-3} \end{cases}$$

$$\sigma_2(\xi) := \begin{cases} \xi^{-2} & \xi > r|p - \widetilde{V}(t)|^{-3} \\ (r|p - \widetilde{V}(t)|^{-3})^{-2} & \xi \leq r|p - \widetilde{V}(t)|^{-3}. \end{cases}$$

Using (4.13), the definition of M_u and the fact that both σ_i are monotonically decreasing, we have the estimate (here $\mathbb{1}_U$ denoted the characteristic function of the set U)

$$\frac{\mathbb{1}_{M_u}(s, x, p)}{|D(s)|^2} \leq \sigma_i(|D(s)|) \leq \sigma_i\left(\frac{1}{4}|p - \widetilde{V}(t)||s - s_0|\right)$$

for $i = 1, 2$ and $s \in [t - \Delta, t]$. This allows us to estimate the integral over M_u by first integrating in time:

$$\begin{aligned} \int_{t-\Delta}^{t} \frac{\mathbb{1}_{M_u}(s, x, p)}{|D(s)|^2}\,\mathrm{d}s &\leq 8|p - \widetilde{V}(t)|^{-1}\int_0^\infty \sigma_i(\xi)\,\mathrm{d}\xi \\ &= 16|p - \widetilde{V}(t)|^{-1}\begin{cases} r^{-1}|p|^3 & i = 1 \\ r^{-1}|p - |\widetilde{V}(t)|^3 & i = 2 \end{cases} \end{aligned}$$

which in turn implies (by taking the minimum of the right hand side)

$$\int_{t-\Delta}^{t} \frac{\mathbb{1}_{M_u}(s,x,p)}{|D(s)|^2} \,\mathrm{d}s \le 16 r^{-1} |p|^2.$$

Therefore we are finally left with

$$\begin{aligned}\iiint_{M_u} \frac{f(t,x,p)}{|\widetilde{X}(s) - X(s;t,x,p)|^2} \,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}s &\le \iint_{\mathbb{R}^6} f(t,x,p) \int_{t-\Delta}^{t} \frac{\mathbb{1}_{M_u}(s,x,p)}{|D(s)|^2} \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}p \\ &\le \frac{c}{r} \iint_{\mathbb{R}^6} |p|^2 f(t,x,p) \,\mathrm{d}x\,\mathrm{d}p \\ &\le \frac{c}{r}\end{aligned}$$

since the kinetic energy is bounded. This concludes the proof. □