3 The Vlasov-Poisson System: Local Existence and Uniqueness

In this section we demonstrate local existence of classical solutions to the Vlasov-Poisson system of equations. This will involve obtaining some *a priori* estimates and an iteration scheme. *A priori* estimates are an essential tool in the analysis of PDEs, and in particular for establishing existence of solutions. We follow [Rein2007].

3.1 Classical Solutions to Vlasov-Poisson: A Rigorous Definition

We start by precisely stating the meaning of a *classical solution*.[17] We recall that the Vlasov-Poisson system is the following system of equations for the unknown $f(t,x,p) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$:

$$\frac{\partial f}{\partial t}(t,x,p) + p \cdot \nabla_x f(t,x,p) + \gamma \mathbf{E}_f(t,x) \cdot \nabla_p f(t,x,p) = 0, \tag{3.1}$$

$$\nabla \cdot \mathbf{E}_f(t,x) = \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \tag{3.2}$$

where $\gamma = +1$ for plasma problems and $\gamma = -1$ for galactic dynamics. We indicate quantities that depend upon f by a corresponding subscript. Alternatively, one could write this system with the field replaced by its potential. Since the potential is only determined up to a constant, one imposes an additional restriction (for instance decay at ∞):

$$\frac{\partial f}{\partial t}(t,x,p) + p \cdot \nabla_x f(t,x,p) - \gamma \nabla \phi_f(t,x) \cdot \nabla_p f(t,x,p) = 0, \tag{3.3}$$

$$-\Delta \phi_f(t,x) = \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \qquad \lim_{|x|\to\infty} \phi_f(t,x) = 0. \tag{3.4}$$

Definition 3.1 (Classical Solution). *A function $f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ is a classical solution of the Vlasov-Poisson system on the interval $I \subset \mathbb{R}$ if:*

- $f \in C^1(I \times \mathbb{R}^3 \times \mathbb{R}^3)$
- *ρ_f and ϕ_f are well-defined, and belong to $C^1(I \times \mathbb{R}^3)$. Moreover, ϕ_f is twice continuously differentiable with respect to x.*
- *For every compact subinterval $J \subset I$, $\mathbf{E}_f = -\nabla \phi_f$ is bounded on $J \times \mathbb{R}^3$.*

Finally, obviously one requires that f satisfy (3.3) *and* (3.4) *on $I \times \mathbb{R}^3 \times \mathbb{R}^3$ and correspondingly that ρ_f and ϕ_f satisfy* (3.3) *and* (3.4) *on $I \times \mathbb{R}^3$.*

Theorem 3.2 (Local Existence of Classical Solutions). *Let $f_0(x,p) \in C^1_0(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f_0 \geq 0$ be given. Then there exists a unique classical solution $f(t,x,p)$ for the system* (3.3)-(3.4) *on some interval $[0,T)$ with $T > 0$ and $f(0,\cdot,\cdot) = f_0$.*

Furthermore, for all $t \in [0,T)$ the function $f(t,\cdot,\cdot)$ is compactly supported and non-negative.

Finally, we have the following breakdown criterion*: if $T > 0$ is chosen to be maximal, and if*

$$\sup_{\substack{(x,p)\in \operatorname{supp} f(t,\cdot,\cdot) \\ t\in[0,T)}} |p| < \infty$$

[17] We shall specialise to the three dimensional classical case.

or

$$\sup_{\substack{x\in\mathbb{R}^3\\ t\in[0,T)}} \rho_f(t,x) < \infty$$

then the solution is global ($T = \infty$).

Remark 3.3. The last part tells us how breakdown of solutions occurs: *both* momenta *and* the particle density must become unbounded. To show global existence (later in the course) one would have to establish *a priori* bounds on these quantities.

Remark 3.4. The assumption that f_0 is compactly supported can be relaxed, to include initial data that decays "sufficiently fast" at infinity (this was done, e.g. by [Horst1981]).

3.2 *A Priori* Estimates

3.2.1 The Free Transport Equation

We start with the basic *free transport equation* which models the force-free transport of particles in the classical case. Letting $f = f(t,x,p)$ with $t \geq 0$, $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $p = \dot{x}$, the initial value problem is

$$\partial_t f + p \cdot \nabla_x f = 0, \qquad f(0,\cdot,\cdot) = f_0. \tag{3.5}$$

We already know that there exists a unique solution for this problem on $[0,\infty)$ (in fact on $(-\infty,\infty)$). Moreover, in this simple case the solution can be written explicitly (the characteristics are trivially $(\dot{X},\dot{V}) = (V,0)$)[18]

$$f(t,x,p) = f_0(x - pt, p)$$

and models particles that move freely (and therefore linearly) without any forces whatsoever acting on them.

Proposition 3.5 (Dispersion). *Let f be the solution to* (3.5) *and assume that $f_0 \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Then:*[19]

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |f(t,x,p)|\,\mathrm{d}p \leq \frac{1}{t^n} \int_{\mathbb{R}^n} \operatorname{ess\,sup}_{q\in\mathbb{R}^n} |f_0(y,q)|\,\mathrm{d}y.$$

In the kinetic case where $f \geq 0$ the density $\rho_f = \int f\,\mathrm{d}p$ decays:

$$\|\rho_f(t,\cdot)\|_\infty \leq \frac{c}{t^n}.$$

3.2.2 The Linear Transport Equation

Now let us consider the linear transport equation

$$\begin{cases} \partial_t u(t,y) + w(t,y) \cdot \nabla_y u(t,y) = 0, & y \in \mathbb{R}^n, t \in (0,T) \\ u(0,y) = u_0(y) \end{cases} \tag{3.6}$$

where $w(t,y) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is given and satisfies, as before,

[18] We use V for the momentum variable characteristic since P will be used later for a different purpose.

[19] For brevity, this is often written as: $\|f(t,\cdot,\cdot)\|_{L^\infty_x(L^1_p)} \leq t^{-n}\|f_0\|_{L^1_x(L^\infty_p)}$

(H1) : $w \in C([0,T] \times \mathbb{R}^n; \mathbb{R}^n)$ and $D_y w \in C([0,T] \times \mathbb{R}^n; \mathrm{M}_n(\mathbb{R}^n))$.

(H2) : $\exists c > 0$ such that $|w(t,y)| \le c(1+|y|)$ for all $(t,y) \in [0,T] \times \mathbb{R}^n$.

Comparing with Vlasov-Poisson, we have:

$$y = (x,p) \in \mathbb{R}^6, \qquad w(t,y) = (p, \gamma \mathbf{E}), \qquad \nabla_y = \nabla_{(x,p)},$$

so that $w \cdot \nabla_y = p \cdot \nabla_x + \gamma \mathbf{E} \cdot \nabla_p$. Of course, the Vlasov-Poisson system is nonlinear (and non-local[20]) since the force depends on f itself. However, it is a common strategy to "forget" this, and imagine that the force is given (then, for instance, *a priori* estimates for the linear transport equation such as Theorem 3.6 below can be used for Vlasov-Poisson). Notice that in any case, the Vlasov flow is divergence-free:

$$\nabla_y \cdot w = \nabla_x \cdot p + \gamma \nabla_p \cdot \mathbf{E} = 0. \tag{3.7}$$

Theorem 3.6 (Properties of the Linear Transport Equation). *Assume that $w(t,y)$ satisfies (H1) and (H2) and that $\nabla_y \cdot w = 0$. Let $u_0 \in C_0^1(\mathbb{R}^n)$. Then the solution u to (3.6) satisfies:*

1. *$u(t,\cdot)$ is compactly supported.*
2. *If $u_0 \ge 0$ then $u(t,\cdot) \ge 0$.*
3. *For all $p \in [1,\infty]$, $\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} = \|u_0\|_{L^p(\mathbb{R}^n)}$.*[21]
4. *For any $\Phi \in C^1(\mathbb{R};\mathbb{R})$ with $\Phi(0) = 0$ we have*

$$\int_{\mathbb{R}^n} \Phi(u(t,y))\,\mathrm{d}y = \int_{\mathbb{R}^n} \Phi(u_0(y))\,\mathrm{d}y, \qquad \forall t \in [0,T]. \tag{3.8}$$

Proof. The first property was already proven in Theorem 2.5, and the second property is an easy consequence of the representation (2.9) of the solution to the linear transport equation using the characteristics. Let us prove (3.8) first (note that this resembles (2.8)). Notice that if u solves the transport equation then so does $\Phi(u)$. This is easily verified by an application of the chain rule. Hence $\Phi(u)$ satisfies the transport equation with initial condition $\Phi(u_0)$. Integrating the transport equation in y we get:

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} \left(\partial_t \Phi(u(t,y)) + w(t,y) \cdot \nabla \Phi(u(t,y))\right) \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \partial_t \Phi(u(t,y))\mathrm{d}y + \int_{\mathbb{R}^n} w(t,y) \cdot \nabla \Phi(u(t,y))\,\mathrm{d}y \\
&= \partial_t \left(\int_{\mathbb{R}^n} \Phi(u(t,y))\mathrm{d}y \right) + \int_{\mathbb{R}^n} \nabla \cdot (w(t,y)\Phi(u(t,y)))\,\mathrm{d}y \\
&= \partial_t \left(\int_{\mathbb{R}^n} \Phi(u(t,y))\,\mathrm{d}y \right),
\end{aligned}$$

where in the third equality we used the fact that $\nabla \cdot w = 0$. This proves Part 4. By letting $\Phi(u) = |u|^p$ for $p \in (1,\infty)$, this also proves conservation of these L^p norms. Note that for $p = 1$ this won't work, as $\Phi(u) = |u|$ isn't C^1. However, in this case we can prove for a smoothed version of $\Phi(u) = |u|$ (i.e. we smooth the singularity at 0) and let the smoothing parameter tend to 0. The details are omitted here.

The fact that the L^∞ norm is conserved is evident from the representation (2.9) and since $u_0 \in C^1$. If u is less smooth then in general the L^∞ norm may decrease. $\square$

[20] This means that the evolution depends on the system as a whole.

[21] One can also consider less smooth initial data in which case this is only true for $p \in [1,\infty)$, and for $p = \infty$ one has $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \le \|u_0\|_{L^\infty(\mathbb{R}^n)}$.

3.2.3 Poisson's Equation

Due to the nature of the nonlinearity in the Vlasov-Poisson system having the form $\nabla\phi_f(t,x)\cdot \nabla_p f(t,x,p)$ we want to obtain some *a priori* estimates on $\nabla\phi_f(t,x)$. We have the following:

Proposition 3.7 (Properties of Solutions to Poisson's Equation). *Given $\rho(x) \in C_0^1(\mathbb{R}^3)$ we define*

$$\phi_\rho(x) := \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\mathrm{d}y.$$

Then:

1. *ϕ_ρ is the unique solution in $C^2(\mathbb{R}^3)$ of $-\Delta\phi = \rho$ with $\lim_{|x|\to\infty}\phi(x) = 0$.*[22]
2. *The force is given by*

   $$\nabla\phi_\rho(x) = -\int \frac{x-y}{|x-y|^3}\rho(y)\,\mathrm{d}y$$

   and we have the decay properties as $|x| \to \infty$

   $$\phi_\rho(x) = O(|x|^{-1}) \quad \textit{and} \quad \nabla\phi_\rho(x) = O(|x|^{-2}).$$

3. *For any $p \in [1,3)$*

   $$\|\nabla\phi_\rho\|_\infty \le c_p\|\rho\|_p^{p/3}\|\rho\|_\infty^{1-p/3} \qquad (c_p \textit{ only depends on } p).$$

4. *For any $p \in [1,3)$, $R > 0$ and $d \in (0,R]$, $\exists c > 0$ independent of ρ, R, d, s.t.*

   $$\|D^2\phi_\rho\|_\infty \le c\left(\frac{\|\rho\|_1}{R^3} + d\|\nabla\rho\|_\infty + (1+\ln(R/d))\|\rho\|_\infty\right),$$
   $$\|D^2\phi_\rho\|_\infty \le c(1+\|\rho\|_\infty)(1+\ln_+\|\nabla\rho\|_\infty) + c\|\rho\|_1.$$

3.3 Sketch of Proof of Local Existence and Uniqueness

The proof of local existence is "standard" in the sense that it follows the ideas outlined in Section 1. However, this does not mean that the proof is easy. This result is due to [Batt1977] and [Ukai1978]. We remind that the system we want to solve is

$$\partial_t f(t,x,p) + p\cdot\nabla_x f(t,x,p) - \gamma\nabla\phi_f(t,x)\cdot\nabla_p f(t,x,p) = 0,$$
$$-\Delta\phi_f(t,x) = \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \qquad \lim_{|x|\to\infty}\phi_f(t,x) = 0.$$

with initial data

$$f(0,\cdot,\cdot) = f_0 \in C_0^1(\mathbb{R}^6).$$

The proof shall follow the following iterative scheme:

Step 0. Set $f^0(t,x,p) = f_0(x,p)$ and define $\phi_{f^0}(t,x)$.

[22] There should be a factor of 4π in Poisson's equation which we omit in accordance with our convention.

Step 1. Let $f^1(t,x,p)$ be the solution to the linear transport equation

$$\partial_t f^1(t,x,p) + p \cdot \nabla_x f^1(t,x,p) - \gamma \nabla \phi_{f^0}(t,x) \cdot \nabla_p f^1(t,x,p) = 0.$$

Define $\phi_{f^1}(t,x)$ which is used to obtain $f^2(t,x,p)$.

... and so on ...

Step N. Let $f^N(t,x,p)$ be the solution to the linear transport equation

$$\partial_t f^N(t,x,p) + p \cdot \nabla_x f^N(t,x,p) - \gamma \nabla \phi_{f^{N-1}}(t,x) \cdot \nabla_p f^N(t,x,p) = 0.$$

Step ∞. Show that as $N \to \infty$, f^N has a C^1 limit, and that this limit satisfies the Vlasov-Poisson system on some time interval $[0,T)$.

Key Ingredients of the Proof. From our study of linear transport equations, we know that the most important ingredient is estimating the vector-field. This amounts to estimating $\nabla \phi_f$ (we drop the superscript N). Define the (crucial!) quantity for $t \in [0,T)$:

$$P(t) := \sup_{(x,p) \in \operatorname{supp} f(t,\cdot,\cdot)} |p|. \tag{3.9}$$

From Theorem 3.6 we know that the norms $\|f(t,\cdot,\cdot)\|_p$ are constant for all $p \in [1,\infty]$; hence

$$\|\rho_f(t,\cdot)\|_\infty = \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} f(t,x,p)\, \mathrm{d}p \le \|f(t,\cdot,\cdot)\|_\infty P^3(t) = cP^3(t)$$

and

$$\|\rho_f(t,\cdot)\|_1 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t,x,p)\, \mathrm{d}p\, \mathrm{d}x = c$$

where c is a constant that may change from line to line. Hence by Proposition 3.7

$$\|\nabla \phi_f(t,\cdot)\|_\infty \le c \|\rho_f(t,\cdot)\|_1^{1/3} \|\rho_f(t,\cdot)\|_\infty^{2/3} \le cP^2(t).$$

The problem therefore reduces to controlling $P(t)$, the maximal momentum. Momentum growth can only happen due to the forcing term in the Vlasov equation: $\nabla \phi_f$. As we have just seen, $\nabla \phi_f$ is controlled by $P^2(t)$, and so we have a typical *Gronwall inequality*:

$$P(t) \le P(0) + c \int_0^t P^2(s)\, \mathrm{d}s.$$

To complete the proof one would have to repeat a similar analysis for derivatives (since we need to show convergence in the C^1-norm). For those, we will show that

$$\|\nabla \rho_f(t,\cdot)\|_\infty \le c \qquad \text{and} \qquad \|D^2 \phi_f(t,\cdot)\|_\infty \le c.$$

3.4 Detailed Proof of Local Existence and Uniqueness

3.4.1 Defining Approximate Solutions

We initiate the problem by iteratively defining a sequence of solutions $(f^N, \rho^N = \rho^{f^N}, \phi^N = \phi^{f^N})$ to an approximated Vlasov-Poisson.

Set

$$f^0(t,x,p) = f_0(x,p)$$

and define

$$\rho^0(t,x) = \int_{\mathbb{R}^3} f^0(t,x,p)\,\mathrm{d}p \quad \text{and} \quad \phi^0(t,x) = \int_{\mathbb{R}^3} \frac{\rho^0(t,z)}{|x-z|}\mathrm{d}z.$$

Suppose that $(f^{N-1}, \rho^{N-1}, \phi^{N-1})$ have been defined and define f^N to be the solution to the linear transport equation

$$\begin{cases} \partial_t f^N(t,x,p) + p \cdot \nabla_x f^N(t,x,p) - \gamma \nabla \phi^{N-1}(t,x) \cdot \nabla_p f^N(t,x,p) = 0, \\ f^N(0,\cdot,\cdot) = f_0. \end{cases}$$

Hence one can write

$$\begin{aligned} f^N(t,x,p) &= f_0(X^{N-1}(0;t,x,p), V^{N-1}(0;t,x,p)), \\ \rho^N(t,x) &= \int_{\mathbb{R}^3} f^N(t,x,p)\,\mathrm{d}p, \\ \phi^N(t,x) &= \int_{\mathbb{R}^3} \frac{\rho^N(t,z)}{|x-z|}\mathrm{d}z. \end{aligned}$$

Goal: show that $\exists T > 0$ such that $\lim_{N\to\infty} f^N$ exists in $C^1([0,T) \times \mathbb{R}^6)$, $\lim_{N\to\infty}(\rho^N, \phi^N)$ exists in $C^1([0,T) \times \mathbb{R}^3)$, that the limits satisfy the Vlasov-Poisson system (uniquely), and that the continuation criterion holds.

3.4.2 The Iterates are Well-Defined

This is a simple proof by induction and is omitted here. One can show that the following holds:

$$\begin{aligned} & f^N(t,x,p) \in C^1([0,\infty) \times \mathbb{R}^6), \\ & \rho^N(t,x) \in C^1([0,\infty) \times \mathbb{R}^3), \\ & \nabla \phi^N(t,x) \in C^1([0,\infty) \times \mathbb{R}^3). \end{aligned}$$

Define $R_0 > 0$ and $P_0 > 0$ to be such that

$$\operatorname{supp} f^0 \subset \{|x| < R_0\} \cap \{|p| < P_0\}$$

and

$$\begin{aligned} P_0(t) &= P_0 \\ P_N(t) &= \sup_{\substack{(x,p)\in \operatorname{supp} f^0(t,\cdot,\cdot) \\ s\in[0,t]}} |V^{N-1}(s;0,x,p)| \end{aligned}$$

Then the following holds:

$$\operatorname{supp} f^N \subset \left\{ |x| < R_0 + \int_0^t P_N(s)\,\mathrm{d}s \right\} \cap \left\{ |p| < P_N(t) \right\}$$
$$\operatorname{supp} \rho^N \subset \left\{ |x| < R_0 + \int_0^t P_N(s)\,\mathrm{d}s \right\}$$
$$\|f^N(t)\|_1 = \|\rho^N(t)\|_1 = \|f^0\|_1$$
$$\|f^N(t)\|_\infty = \|f^0\|_\infty$$
$$\|\rho^N(t)\|_\infty \le c\|f^0\|_\infty P_N^3(t)$$
$$\|\nabla\phi^N(t)\|_\infty \le C(f^0) P_N^2(t)$$

where by Proposition 3.7

$$C(f^0) = \|f^0\|_1^{1/3}\|f^0\|_\infty^{2/3} \tag{3.10}$$

up to some multiplicative constant.

3.4.3 A Uniform Bound for the Maximal Momentum

Intuitively, we know that for each N, the particle acceleration is given by $\nabla\phi^N(t)$ for which we have the bound $C(f^0)P_N^2(t)$. This suggests that all momenta can be uniformly bounded as follows. Let $\delta > 0$ and $P : [0,\delta) \to (0,\infty)$ be such that P is the maximal solution of

$$P(t) = P_0 + C(f^0)\int_0^t P^2(s)\,\mathrm{d}s \qquad \text{i.e. } P(t) = \frac{P_0}{1 - P_0 C(f^0)t} \text{ and } \delta = \frac{1}{P_0 C(f^0)}.$$

Claim. We have the uniform bound:

$$P_N(t) \le P(t), \qquad \forall N \ge 0, t \in [0,\delta).$$

Assuming this for the moment, we immediately have for all $N \ge 0$ and $t \in [0,\delta)$:

$$\|\rho^N(t,\cdot)\|_\infty \le c\|f^0\|_\infty P^3(t),$$
$$\|\nabla\phi^N(t,\cdot)\|_\infty \le C(f^0)P^2(t).$$

Proof of claim. The claim is clearly true for $N = 0$. Hence we assume it is true for N and prove for $N+1$. For any $0 \le s \le t < \delta$ and $(x,p) \in \operatorname{supp} f^0$:

$$\begin{aligned} |V^N(s;0,x,p)| &\le |p| + \int_0^s \|\nabla\phi^N(\tau,\cdot)\|_\infty \mathrm{d}\tau \\ &\le P_0 + C(f^0)\int_0^s P_N^2(\tau)\,\mathrm{d}\tau \\ &\le P_0 + C(f^0)\int_0^t P^2(\tau)\,\mathrm{d}\tau = P(t). \end{aligned}$$

3.4.4 A Uniform Bound for $\nabla\rho^N$ and $D^2\phi^N$

For C^1 convergence we need we need uniform convergence of derivatives of ρ^N and $\nabla\phi^N$ on subintervals of $[0,\delta)$. Hence we let $\delta_0 \in (0,\delta)$ and claim the following:

Claim. $\exists c = c(f^0,\delta_0) > 0$ such that

$$\|\nabla\rho^N(t,\cdot)\|_\infty + \|D^2\phi^N(t,\cdot)\|_\infty \le c, \qquad \forall t \in [0,\delta_0],\ N \ge 0.$$

To show this, we first claim:

Sub-Claim 1. We can estimate

$$\|\nabla\rho^{N+1}(t,\cdot)\|_\infty \le c\exp\left[\int_0^t \|D^2\phi^N(\tau,\cdot)\|_\infty \mathrm{d}\tau\right], \qquad 0\le t\le\delta_0.$$

Assuming this for the moment, we prove the claim:

Proof of Claim. Recall the estimate on $\|D^2\phi^N(t,\cdot)\|_\infty$ from Proposition 3.7:

$$\|D^2\phi^{N+1}(t,\cdot)\|_\infty \le c(1+\|\rho^{N+1}(t,\cdot)\|_\infty)(1+\ln_+\|\nabla\rho^{N+1}(t,\cdot)\|_\infty)+c\|\rho^{N+1}(t,\cdot)\|_1.$$

Using Sub-Claim 1 together with the estimate for $\|\rho^N(t,\cdot)\|_\infty$ that we obtained in the previous step, we have:

$$\begin{aligned}\|D^2\phi^{N+1}(t,\cdot)\|_\infty &\le c(1+\|\rho^{N+1}(t,\cdot)\|_\infty)(1+\ln_+\|\nabla\rho^{N+1}(t,\cdot)\|_\infty)+c\|\rho^{N+1}(t,\cdot)\|_1\\ &\le c(1+\|f^0\|_\infty P^3(t))\left(1+\ln_+\exp\left[\int_0^t\|D^2\phi^N(\tau,\cdot)\|_\infty\mathrm{d}\tau\right]\right)+c\\ &\le c\left(1+\int_0^t\|D^2\phi^N(\tau,\cdot)\|_\infty\mathrm{d}\tau\right).\end{aligned}$$

Hence by induction

$$\|D^2\phi^N(t,\cdot)\|_\infty \le ce^{ct}, \qquad \forall t\in[0,\delta_0], N\ge 0$$

(here we assumed that c is so large that $\|D^2\phi^0(t,\cdot)\|_\infty\le c$). □

Now we are left with proving the sub-claim:

Proof of Sub-Claim 1. We first note that

$$\rho^{N+1}(t,x)=\int_{\mathbb{R}^3} f^{N+1}(t,x,p)\,\mathrm{d}p=\int_{\mathbb{R}^3} f_0(X^N(0;t,x,p),V^N(0;t,x,p))\,\mathrm{d}p.$$

Hence

$$\begin{aligned}|\nabla\rho^{N+1}(t,x)| &\le \int_{|p|\le P(t)}\left|\nabla_x\left(f_0(X^N(0;t,x,p),V^N(0;t,x,p))\right)\right|\,\mathrm{d}p\\ &\le c\left(\|\nabla_x X^N(0;t,\cdot,\cdot)\|_\infty+\|\nabla_x V^N(0;t,\cdot,\cdot)\|_\infty\right).\end{aligned}$$

From Sub-Claim 2 below, we know that

$$|\nabla_x X^N(s;t,x,p)|+|\nabla_x V^N(s;t,x,p)|\le \\ 1+\int_s^t\left(1+\|D^2\phi^N(\tau,\cdot)\|_\infty\right)\left(|\nabla_x X^N(\tau,t,x,p)|+|\nabla_x V^N(\tau,t,x,p)|\right)\,\mathrm{d}\tau.$$

So *Gronwall's inequality* leads to

$$|\nabla_x X^N(s;t,x,p)|+|\nabla_x V^N(s;t,x,p)|\le\exp\left[\int_0^t\left(1+\|D^2\phi^N(\tau,\cdot)\|_\infty\right)\,\mathrm{d}\tau\right]$$

which completes the proof of the sub-claim. □

Sub-Claim 2. We claim that for all $(x,p)\in\mathbb{R}^6$ and $s,t\in[0,\delta_0]$,

$$|\nabla_x X^N(s;t,x,p)|+|\nabla_x V^N(s;t,x,p)|\le \\ 1+\int_s^t\left(1+\|D^2\phi^N(\tau,\cdot)\|_\infty\right)\left(|\nabla_x X^N(\tau,t,x,p)|+|\nabla_x V^N(\tau,t,x,p)|\right)\,\mathrm{d}\tau.$$

Proof of Sub-Claim 2. Exercise. □

3.4.5 The Sequence $\{f^N\}$ Has a Limit

Since $f^N(t,x,p) = f_0(X^{N-1}(0;t,x,p), V^{N-1}(0;t,x,p))$ we have

$$\begin{aligned}
&|f^{N+1}(t,x,p) - f^N(t,x,p)| \\
&\qquad = |f_0(X^N(0;t,x,p), V^N(0;t,x,p)) - f_0(X^{N-1}(0;t,x,p), V^{N-1}(0;t,x,p))| \\
&\qquad \le c\left(|X^N(0;t,x,p) - X^{N-1}(0;t,x,p)| + |V^N(0;t,x,p) - V^{N-1}(0;t,x,p)|\right).
\end{aligned}$$

Claim. The following estimate holds:

$$\begin{aligned}
&|X^N(s;t,x,p) - X^{N-1}(s;t,x,p)| + |V^N(s;t,x,p) - V^{N-1}(s;t,x,p)| \\
&\qquad\qquad \le c\int_0^t \|f^N(\tau,\cdot,\cdot) - f^{N-1}(\tau,\cdot,\cdot)\|_\infty \,\mathrm{d}\tau.
\end{aligned}$$

Assuming this claim, we then have

$$\|f^{N+1}(t,\cdot,\cdot) - f^N(t,\cdot,\cdot)\|_\infty \le c\int_0^t \|f^N(\tau,\cdot,\cdot) - f^{N-1}(\tau,\cdot,\cdot)\|_\infty \,\mathrm{d}\tau$$

which yields

$$\|f^{N+1}(t,\cdot,\cdot) - f^N(t,\cdot,\cdot)\|_\infty \le ct^N(N!)^{-1}, \qquad n \ge 0, t \in [0,\delta_0].$$

Hence the sequence $\{f^N\}$ is uniformly Cauchy and converges uniformly on $[0,\delta_0] \times \mathbb{R}^6$ to some function $f \in C([0,\delta_0] \times \mathbb{R}^6)$.

3.4.6 Properties of the Limit

ρ^N and ϕ^N also converge uniformly:

$$\rho^N \to \rho_f,\ \phi^N \to \phi_f, \qquad \text{uniformly on } [0,\delta_0] \times \mathbb{R}^3.$$

The support of f satisfies:

$$\operatorname{supp} f \subset \left\{|x| < R_0 + \int_0^t P(s)\,\mathrm{d}s\right\} \cap \left\{|p| < P(t)\right\}.$$

The estimates on on Poisson's equation (Proposition 3.7) lead to (exercise):

$$\phi_f, \nabla\phi_f, D^2\phi_f \in C([0,\delta_0] \times \mathbb{R}^3).$$

This implies that

$$(X^N, V^N) \to (X, V) \in C^1([0,\delta_0] \times [0,\delta_0] \times \mathbb{R}^6; \mathbb{R}^6)$$

and this is the flow due to the limiting field $\nabla\phi_f$. Hence

$$f(t,x,p) = \lim_{N\to\infty} f_0(X^N(0;t,x,p), V^N(0;t,x,p)) = f_0(X(0;t,x,p), V(0;t,x,p))$$

and $f \in C^1([0,\delta_0] \times \mathbb{R}^6)$.

3.4.7 Uniqueness

Uniqueness is a simple consequence of *Gronwall's inequality.*

3.4.8 Proof of the Continuation Criterion

This is a proof by contradiction. Assume that the maximal time interval on which the solution f can defined is $[0,T)$, with $T < \infty$, but that neither $\operatorname{supp}|p|$ nor $\|\rho_f\|_\infty$ blowup as $t \nearrow T$. Take some $T_0 = T - \varepsilon$ (ε to be chosen sufficiently small) and restart the problem from T_0, showing that it is possible to go beyond T.